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Error analysis of interpolating element free Galerkin method to solve non-linear extended Fisher–Kolmogorov equation

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ABSTRACT

Nonlinear partial differential equations (PDEs) play an important role in the modeling of the natural phenomena as they have great significance in real-world applications. This investigation proposes a new algorithm to find the numerical solution of the non-linear extended Fisher–Kolmogorov equation. Firstly, the time variable is discretized by a second-order finite difference scheme. The rate of convergence and stability of the semi-discrete formulation are studied by the energy method. The existence and uniqueness of the solution of the weak form based on the proposed technique have been proved in detail. Furthermore, the interpolating element free Galerkin approach based on the interpolation moving least-squares approximation is employed to derive a fully discrete scheme. Finally, the error estimate of the full-discrete plan is proposed and its convergence order is $O(\tau^2+\delta^{m+1})$ in which τ, δ and m denote the time step, the radius of the weight function and smoothness of the exact solution of the main problem, respectively.

1. Introduction

Most real-world problems may be modeled by the nonlinear PDEs. On the other hand, some of them cannot be solved by analytical methods [1,2]. Also, numerical methods such as the finite difference, finite element or finite volume methods have been applied to solve these problems. But each of numerical or analytical procedures have pros and cons.

The nonlinear extended Fisher–Kolmogorov equation is assumed as follows [3,4]

$$\begin{cases} \dfrac{\partial u}{\partial t} + \kappa_1 \Delta^2 u - \kappa_2 \Delta u = F(u), & \text{in } \Omega, \\ u = f_1, \qquad \Delta u = f_2, & \text{on } \partial\Omega, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

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where κ_1 and κ_2 are positive constants, f_1 and f_2 are given functions, $F(u) = u - u^3$ and also $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Furthermore, the nonlinear term $F(u)$ satisfies the Lipschitz condition according to the following lemma.

Lemma 1.1. *Let $u \in C^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^d$ be a bounded closed set. Then the nonlinear term $F(u) = u - u^3$ in Eq. (1.1) satisfies the inequality*

$$|F(u_1) - F(u_2)| \leq L\,|u_1 - u_2|\,.$$

Proof. According to the assumption, we have

$$\min\left\{u \;:\; u \in C^\infty(\Omega)\right\} \leq u \leq \max\left\{u \;:\; u \in C^\infty(\Omega)\right\},$$

and thus there exists a positive integer M such that

$$\max_{u\in C^\infty(\Omega)} |u| \leq M.$$

Now, we have

$$\begin{aligned}
|F(u_1) - F(u_2)| &= \left|u_1 - u_1^3 - u_2 + u_2^3\right| = \left|(u_2^3 - u_1^3) - (u_2 - u_1)\right| \\
&\leq \left|u_2^3 - u_1^3\right| + |u_2 - u_1| = |u_2 - u_1|\left|u_2^2 + u_2u_1 + u_1^2\right| + |u_2 - u_1| \\
&= |u_2 - u_1|\left(\left|u_2^2 + u_2u_1 + u_1^2\right| + 1\right) \leq |u_2 - u_1|\left(|u_2|^2 + |u_2|\,|u_1| + |u_1|^2 + 1\right) \\
&\leq (3M^2 + 1)\,|u_2 - u_1| \leq L\,|u_2 - u_1|\,,
\end{aligned}$$

which completes the proof. ■

Authors of [3,4] presented a Crank–Nicolson finite difference scheme to solve the one- and two-dimensional nonlinear evolutionary extended Fisher–Kolmogorov equations. Authors of [5] developed conservative finite difference schemes with error analysis for the 2D Rosenau-KdV equation. The finite element Galerkin method is proposed in [6] for solving the Rosenau-RLW (RRLW) equation [7–9] and the existence and uniqueness of the weak solution are studied. The existence and uniqueness of numerical solutions of the KdV like Rosenau equation [9] are studied in [10].

A conservative difference scheme is proposed in [11,12] to the numerical study of the generalized Rosenau–Kawahara–RLW equation. The main aim of [13] is to develop a second-order splitting combined with orthogonal cubic spline collocation method to solve the extended Fisher–Kolmogorov equation. Existence, uniqueness and regularity solution of the extended Fisher–Kolmogorov equation have been studied in [14] based on the C^1-conforming finite element method. The quintic B-spline collocation method has been used in [15] to find the numerical solution of the extended Fisher–Kolmogorov equation. The Fisher–Kolmogorov equation was solved in [16] by using a differential quadrature technique based upon the quintic B-spline. Authors of [17] solved the nonlinear PDEs based on a collocation method with three different bases such as B-spline, Fourier and Chebyshev. The local boundary integral equation (LBIE) method based on generalized moving least squares (GMLS) is developed in [18] for solving extended Fisher—Kolmogorov (EFK) equation.

The main aim of [19] is to develop a numerical solution based on a wavelet collocation method which uses Chebyshev wavelets for solving the two-space-dimensional nonlinear Fisher–Kolmogorov–Petrovsky–Piscounov (Fisher–KPP) equation and the two-space-dimensional nonlinear fourth-order extended Fisher–Kolmogorov (EFK) equation. The author of [20] developed a non-uniform Haar wavelet based on collocation method for solving two-dimensional convection dominated equations and two-dimensional nearly singular elliptic partial differential equations. The author of [21] developed a numerical solution of 3D convection–diffusion problems for high Reynolds numbers and variable coefficients via a meshless method based on a polynomial basis. The interpolating element-free Galerkin technique is developed in [22] for solving 2D generalized Benjamin-Bona-Mahony-Burgers and regularized long-wave equations on non-rectangular domains. The main aim of [23] is to combine the two-grid procedure with the element free Galerkin method for solving Rosenau-regularized long wave (RRLW) equation.

Belytschko and his co-workers have introduced the EFG method [24–26] as a generalization of the finite element method. A new development of the EFG method is the interpolating EFG procedure which is based upon the interpolating moving least-squares approximation introduced by Lancaster and Salkauskas [27]. In recent decades, several researchers are working on this field and they have solved some important models. Authors of [28,29] presented a simple way to build the shape functions of IMLS for solving 2D potential and elasticity models by a new boundary element-free method. Authors of [30,31] employed improved IMLS approximation to the numerical solution of 2D elasticity and potential problems. Wang and his co-authors [32] have proved an error estimation for the IMLS procedure. Sun et al. [33] investigated an error estimation for the IMLS approximation and its derivative. Meng and his co-workers proposed the dimension split EFG method to solve 3D potential problems. Li [34] studied stability and convergence of the MLS approximation and also he developed a new idea to treat the instability of the MLS approximation. Authors of [35] developed a Crank–Nicolson method and interpolating stabilized element free Galerkin method for solving Oldroyd model as a generalized incompressible Navier-Stokes equation. An upwind local radial basis functions-differential quadrature (RBFs-DQ) technique is proposed in [36] to simulate the shallow water equation. The main aim of [37] is to develop a numerical scheme based on the reproducing kernel particle Petrov–Galerkin method for solving two-dimensional nonstationary incompressible Boussinesq equations. Authors of [38] developed a meshless numerical procedure based on the interpolating element free Galerkin method to simulate the groundwater equation (GWE). The main aim of [39] is to develop a new version of the element-free Galerkin method based on the shape functions of reproducing kernel particle method (RKPM) for solving 2D fractional Tricomi-type equation with Robin boundary condition. Authors of [40] introduced the complex variable interpolating moving least-squares method. A direct meshless local collocation method is proposed in [41] for solving stochastic Cahn-Hilliard-Cook and stochastic Swift-Hohenberg equations.

The IMLS technique has been used to solve some problems such as 2D elastoplasticity models [42], nonlinear elastic models [43], the polymer gels [44], 2D elastoplasticity problems [42,45], 2D large deformation problems [46], incompressible Navier–Stokes equation [47,48], 2D potential problems [49,50], 2D large deformation of swelling of gels [51], etc.

1.1. The structure of this paper

In this paper, we apply the new technique for solving the introduced model. The outline of this paper is as follows. In Section 2, we propose some notations and required lemma for the error analysis. In Section 3, the IMLS approximation will be explained. In Section 4, the stability and convergence of the time-discrete scheme will be studied. The existence and uniqueness of the solution of the Galerkin weak form are investigated in Section 5. Error estimate of the fully-discrete scheme is proved in Section 6. In Section 7 some numerical experiments are reported. Finally, the conclusions are given in Section 8.

2. Some notations, preliminaries and associated weak form

To construct the weak form of the main model, we need some notations. Let $\Omega \in \mathbb{R}^d$ be a bounded and open set ($d = 2$ or 3). For $p < +\infty$, $L^p(\Omega)$ denotes the space of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |u|^p d\Omega \le +\infty.$$

The inner product and associated norm $L^p(\Omega)$ are defined as [52]

$$(u, v) = \int_\Omega uv d\Omega, \qquad \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p d\Omega\right)^{\frac{1}{p}}.$$

Assume $\beta = (\beta_1, \ldots, \beta_d)$ and $|\beta| = \sum_{i=1}^{d} \beta_i$, and

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}.$$

We define the Hilbert space as

$$H^m(\Omega) = \left\{ v \in L^2(\Omega), \quad D^\alpha v \in L^2(\Omega) \ \textit{for all} \ |\alpha| \le m \right\},$$

with the inner product

$$\langle u, v\rangle_m = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v d\Omega,$$

and norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)} \right).$$

Proposition 2.1 (*Pages 13 and 14 of [53]*). *Let θ_n, ϕ_n and δ_k be non-negative sequences and $c_0 > 0$, and suppose that the sequence θ_n satisfies*

$$\begin{cases} \theta_0 \le c_0, \\ \theta_n \le c_0 + \sum_{k=0}^{n-1} \delta_k + \sum_{k=0}^{n-1} \phi_k \theta_k, & n \ge 1. \end{cases}$$

If $c_0 \geq 0$ and $\delta_0 \geq 0$, it follows that

$$\theta_n \leq \left(c_0 + \sum_{k=0}^{n-1} \delta_k\right) \exp\left(\sum_{k=0}^{n-1} \phi_k\right), \qquad n \geq 1.$$

3. Interpolating moving least squares (IMLS) approximation

We put $\boldsymbol{\zeta} = \left\{\boldsymbol{\zeta}_i\right\}_{i=1}^{N}$ as a scattered data in the domain $\Omega \subset \mathbb{R}^n$. The influence domain of node $\boldsymbol{\zeta}$ with size δ is [54]

$$\wp(\boldsymbol{\zeta}) = \left\{\boldsymbol{\zeta}^* \in \mathbb{R}^n \quad : \quad \left\|\boldsymbol{\zeta} - \boldsymbol{\zeta}^*\right\|_{L^2(\Omega)} < \delta(\boldsymbol{\zeta})\right\}. \tag{3.1}$$

Furthermore, the influence domain for $\boldsymbol{\zeta}_i$ is

$$\wp_i = \left\{\boldsymbol{\zeta}^* \in R^n \quad : \quad \left\|\boldsymbol{\zeta}_i - \boldsymbol{\zeta}^*\right\|_{L^2(\Omega)} < \delta_i\right\}. \tag{3.2}$$

We define

$$w_i(x) = \begin{cases} \varphi\left(\frac{\|\zeta - \zeta_i\|_2}{\delta_i}\right) \left\|\frac{\zeta - \zeta_i}{\delta_i}\right\|_{L^2(\Omega)}^{-\varpi}, & x \in \wp_i, \\ 0, & x \notin \wp_i, \end{cases} \tag{3.3}$$

for $i = 1, 2, \ldots, N$ and

$$\mathbf{p}(\boldsymbol{\zeta}) = [p_0(\boldsymbol{\zeta}), p_1(\boldsymbol{\zeta}), \ldots, p_m(\boldsymbol{\zeta})]^T, \qquad \boldsymbol{\zeta} \in \Omega, \tag{3.4}$$

is a set of polynomial basis of degree m. Consider $span = (p_0(\boldsymbol{\zeta}), p_1(\boldsymbol{\zeta}), \ldots, p_m(\boldsymbol{\zeta}))$ and [54]

$$q_0(\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}}) = \frac{p_0(\boldsymbol{\zeta})}{(p_0, p_0)_x^{\frac{1}{2}}} = \frac{1}{\left[\sum_{i \in \mathcal{E}(\boldsymbol{\zeta})} w_i(\boldsymbol{\zeta})\right]^{\frac{1}{2}}}, \tag{3.5}$$

where

$$(f, g)_{\zeta} = \sum_{i \in \mathcal{E}(\boldsymbol{\zeta})} w_i(\boldsymbol{\zeta}) f(\boldsymbol{\zeta}_i) g(\boldsymbol{\zeta}_i). \tag{3.6}$$

We set

$$q_i(\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}}) = p_i(\bar{\boldsymbol{\zeta}}) - \sum_{l \in \mathcal{E}(\boldsymbol{\zeta})} v_l(\boldsymbol{\zeta}) p_i(\boldsymbol{\zeta}_l), \tag{3.7}$$

where

$$v_l(\boldsymbol{\zeta}) = \frac{w_l(\boldsymbol{\zeta})}{\sum_{j \in \mathcal{E}(\boldsymbol{\zeta})} w_j(\boldsymbol{\zeta})}. \tag{3.8}$$

Let u_h be an approximation of the known function u. Thus, the approximate solution of u_h will be [54]

$$u_h(\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}}) = \sum_{i=0}^{m} q_i(\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}}) a_i(\boldsymbol{\zeta}) = q_0(\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}}) a_0(\boldsymbol{\zeta}) + \mathbf{q}^{\mathrm{T}}(\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}}) \mathbf{a}(\boldsymbol{\zeta}), \tag{3.9}$$

where the unknown coefficients $a_i(\boldsymbol{\zeta})$ can be calculated by minimizing the discrete L^2-norm

$$J(x) = \sum_{i \in \mathcal{E}(\boldsymbol{\zeta})} w_i(\boldsymbol{\zeta}) \left[u(\boldsymbol{\zeta}_i) - u_h(\boldsymbol{\zeta}, \boldsymbol{\zeta}_i)\right]^2 = \sum_{i \in \mathcal{E}(\boldsymbol{\zeta})} w_i(\boldsymbol{\zeta}) \left[u(\boldsymbol{\zeta}_i) - \sum_{i=0}^{m} q_i(\boldsymbol{\zeta}, \boldsymbol{\zeta}_i) a_i(\boldsymbol{\zeta})\right]^2. \tag{3.10}$$

From relation (3.6), Eq. (3.10) will be

$$(u(\cdot) - u_h(\boldsymbol{\zeta}, \cdot), q_i(\boldsymbol{\zeta}, \cdot))_{\zeta} = 0, \qquad i = 0, 1, 2, \ldots, m. \tag{3.11}$$

According to the orthogonality of q_i we write [54]

$$a_0(\boldsymbol{\zeta}) = (u, q_0(\boldsymbol{\zeta}, \cdot))_{\zeta}, \tag{3.12}$$

$$\sum_{i=1}^{m} (q_i(\zeta,\cdot), q_j(\zeta,\cdot))_{\zeta} a_i(\zeta) = (u, q_j(\zeta,\cdot))_{\zeta}, \qquad j = 1, 2, \ldots, m. \tag{3.13}$$

Now, we obtain Eq. (3.13) as

$$\mathbf{A}(\zeta)\mathbf{a}(\zeta) = \mathbf{B}(\zeta)\mathbf{u}, \tag{3.14}$$

where

$$\mathbf{u} = \left[u(\zeta_{I_1}), u(\zeta_{I_2}), \ldots, u(\zeta_{I_{\delta(\zeta)}})\right]^T \mathbf{A}(\zeta) = \mathbf{B}(\zeta)\mathbf{Q}(\zeta), \tag{3.15}$$

$$\mathbf{Q}(\zeta) = \left[\mathbf{q}(\zeta, \zeta_{I_1}), \mathbf{q}(\zeta, \zeta_{I_2}), \ldots \mathbf{q}(\zeta, \zeta_{I_{\delta(\zeta)}})\right], \tag{3.16}$$

and also [54]

$$\mathbf{B}_{ij}(\zeta) = \begin{cases} w_{I_j}(\zeta) q_i(\zeta, \zeta_{I_j}), & \zeta \neq \zeta_{I_j}, \\ \displaystyle\sum_{k \in E(x), k \neq j} w_k(\zeta) \left[p_i(\zeta_{I_j}) - p_i(\zeta_k)\right], & x = \zeta_{I_j}. \end{cases} \tag{3.17}$$

Eq. (3.14), gives

$$\mathbf{a}(\zeta) = \mathbf{A}^{-1}(\zeta)\mathbf{B}(\zeta)\mathbf{u}. \tag{3.18}$$

Pursuant to the previous explanations, we get [54]

$$q_0(\zeta, \overline{\zeta}) a_0(\zeta) = q_0(\zeta, \overline{\zeta})(u, q_0(\zeta,\cdot))_{\zeta} = \sum_{i \in \mathcal{E}(\zeta)} v_i(\zeta) u(\zeta_i) = \mathbf{v}^{\mathrm{T}}(\zeta)\mathbf{u}, \tag{3.19}$$

as

$$\mathbf{v}(\zeta) = \left[v_{I_1}(\zeta), v_{I_2}(\zeta), \ldots, v_{I_{\delta(\zeta)}}(\zeta)\right]^{\mathrm{T}}. \tag{3.20}$$

Substituting Eqs. (3.18) and (3.19) into Eq. (3.9), results

$$u_h(\zeta, \overline{\zeta}) = \mathbf{v}^{\mathrm{T}}(\zeta) u + \mathbf{q}^{\mathrm{T}}(\zeta, \overline{\zeta}) \mathbf{A}^{-1}(\zeta)\mathbf{B}(\zeta)\mathbf{u}. \tag{3.21}$$

However, the approximate solution is

$$u(\zeta) \approx u_h(\zeta) = u_h(\zeta, \overline{\zeta})\big|_{\overline{\zeta}=\zeta} = \left[\mathbf{v}^{\mathrm{T}}(\zeta) + \mathbf{q}^{\mathrm{T}}(\zeta, \overline{\zeta})\mathbf{A}^{-1}(\zeta)\mathbf{B}(\zeta)\right]\mathbf{u}. \tag{3.22}$$

Finally, the IMLS shape functions are [54]

$$\phi_i(\zeta) = \begin{cases} v_i(\zeta) + \displaystyle\sum_{j=1}^{m} q_j(\zeta, \zeta)\left[\mathbf{A}^{-1}(\zeta)\mathbf{B}(\zeta)\right]_{jk}, & i = I_k \in \mathcal{E}(\zeta), \\ 0, & i \notin \mathcal{E}(\zeta). \end{cases} \tag{3.23}$$

4. Stability and convergence of the time-discrete scheme

We define $t_n = n\tau$ for $n = 0, 1, \ldots, N$ where $\tau = T/N$. The weak form is

$$\begin{cases} \langle u_t, v\rangle + \kappa_1 \langle \Delta u, \Delta v\rangle + \kappa_2 \langle \nabla u, \nabla v\rangle = \langle F(u), v\rangle, & \forall v \in H_0^2(\Omega), \\ u(0) = u_0. \end{cases} \tag{4.1}$$

Now, the discrete weak form is

$$\langle u_{ht}, v_h\rangle + \kappa_1 \langle \Delta u_h, \Delta v_h\rangle + \kappa_2 \langle \nabla u_h, \nabla v_h\rangle = \langle F(u_h), v_h\rangle, \qquad \forall v_h \in V_h, \tag{4.2}$$

where V_h is a finite-dimensional subspace of $H_0^2(\Omega)$. With regard to notations

$$\overline{\delta}_t U^n = \frac{U^n - U^{n-1}}{\tau}, \qquad \overline{U}^n = \frac{1}{2}\left(U^n + U^{n-1}\right),$$

the full-discrete scheme is

$$\left\langle \overline{\delta}_t U_h^n, v_h\right\rangle + \kappa_1 \left\langle \Delta \overline{U}_h^n, \Delta v_h\right\rangle + \kappa_2 \left\langle \nabla \overline{U}_h^n, \nabla v_h\right\rangle = \left\langle F(\overline{U}_h^n), v_h\right\rangle, \qquad \forall v_h \in V_h, \tag{4.3}$$

in which U_h^n is an approximation of u_h^n. This section involves analyzing stability and convergence order of the time-discrete formulation.

Theorem 4.1. *Let $U_h^n \in H_0^2(\Omega)$. Then the numerical procedure (4.3) is unconditionally stable.*

Proof. The roundoff error is

$$\left\langle \bar{\delta}_t \mathfrak{X}_h^n, v_h \right\rangle + \left\langle \Delta \overline{\mathfrak{X}}_h^n, \Delta v_h \right\rangle + \left\langle \nabla \overline{\mathfrak{X}}_h^n, \nabla v_h \right\rangle = \left\langle F(\bar{U}_h^n) - F(\bar{U}_h^{n*}), v_h \right\rangle, \qquad \forall v_h \in J_h, \tag{4.4}$$

where

$$\overline{\mathfrak{X}}_h^n = \overline{U}_h^n - \overline{U}_h^{n*}, \tag{4.5}$$

and $\overline{U}_h^{n*}$ is the approximation of $\bar{U}_h^n$. Substituting $v_h = \overline{\mathfrak{X}}_h^n$ in Eq. (4.4) yields

$$\begin{aligned} \frac{1}{2\tau}\left[\left\|\mathfrak{X}_h^n\right\|_{L^2}^2 - \left\|\mathfrak{X}_h^{n-1}\right\|_{L^2}^2\right] + \frac{1}{2}\left[\kappa_1 \left\|\Delta \overline{\mathfrak{X}}_H^n\right\|_{L^2}^2 + \kappa_1 \left\|\Delta \overline{\mathfrak{X}}_h^{n-1}\right\|_{L^2}^2 + \kappa_2 \left\|\nabla \overline{\mathfrak{X}}_h^n\right\|_{L^2}^2 + \kappa_2 \left\|\nabla \overline{\mathfrak{X}}_h^{n-1}\right\|_{L^2}^2\right] \\ \le L \left\|\overline{\mathfrak{X}}_h^n\right\|_{L^2} \left\|\overline{\mathfrak{X}}_h^n\right\|_{L^2}. \end{aligned} \tag{4.6}$$

Changing index n to i and then summing i from 1 to n result in

$$\begin{aligned} \frac{1}{2\tau}\sum_{i=1}^{n}\left[\left\|\mathfrak{X}_h^i\right\|_{L^2}^2 - \left\|\mathfrak{X}_h^{i-1}\right\|_{L^2}^2\right] + \frac{1}{2}\sum_{i=1}^{n}\left[\kappa_1 \left\|\Delta \overline{\mathfrak{X}}_h^i\right\|_{L^2}^2 + \kappa_1 \left\|\Delta \overline{\mathfrak{X}}_h^{i-1}\right\|_{L^2}^2\right] \\ + \frac{1}{2}\sum_{i=1}^{n}\left[\kappa_2 \left\|\nabla \overline{\mathfrak{X}}_h^i\right\|_{L^2}^2 + \kappa_2 \left\|\nabla \overline{\mathfrak{X}}_h^{i-1}\right\|_{L^2}^2\right] \le L \sum_{i=1}^{n} \left\|\overline{\mathfrak{X}}_h^i\right\|_{L^2}^2. \end{aligned} \tag{4.7}$$

Simplifying the above relations outputs

$$\begin{aligned} \left[\left\|\mathfrak{X}_h^n\right\|_{L^2}^2 - \left\|\mathfrak{X}_h^0\right\|_{L^2}^2\right] + \tau \sum_{i=1}^{n}\left[\kappa_1 \left\|\Delta \overline{\mathfrak{X}}_h^i\right\|_{L^2}^2 + \kappa_1 \left\|\Delta \overline{\mathfrak{X}}_h^{i-1}\right\|_{L^2}^2\right] \\ + \tau \sum_{i=1}^{n}\left[\kappa_2 \left\|\nabla \overline{\mathfrak{X}}_h^i\right\|_{L^2}^2 + \kappa_2 \left\|\nabla \overline{\mathfrak{X}}_h^{i-1}\right\|_{L^2}^2\right] \le 2L\tau \sum_{i=1}^{n} \left\|\overline{\mathfrak{X}}_h^i\right\|_{L^2}^2, \end{aligned} \tag{4.8}$$

or equivalently

$$\left\|\mathfrak{X}_h^n\right\|_{L^2}^2 \le \left\|\mathfrak{X}_h^0\right\|_{L^2}^2 + 2L\tau \sum_{i=1}^{n} \left\|\mathfrak{X}_h^i\right\|_{L^2}^2. \tag{4.9}$$

Applying Proposition 2.1 gives

$$\begin{aligned} \left\|\mathfrak{X}_h^n\right\|_{L^2(\Omega)}^2 &\le \left\|\mathfrak{X}_h^0\right\|_{L^2(\Omega)}^2 \exp\left(\sum_{i=0}^{n} 2\tau L C_\Omega\right), \\ &\le \left\|\mathfrak{X}_h^0\right\|_{L^2(\Omega)}^2 \exp\left(T L C_\Omega\right) = C_* \left\|\mathfrak{X}_h^0\right\|_{L^2(\Omega)}^2, \end{aligned} \tag{4.10}$$

thus

$$\left\|\mathfrak{X}_h^n\right\|_{L^2(\Omega)}^2 \le C_* \left\|\mathfrak{X}_h^0\right\|_{L^2(\Omega)}^2, \tag{4.11}$$

which completes the proof. ■

Theorem 4.2. *Let u_h^n and U_h^n be belong to $H_0^2(\Omega)$. Then, the numerical procedure (4.3) is convergent and the convergence order is $\mathcal{O}(\tau^2)$.*

Proof. We have

$$\left\langle \bar{\delta}_t u_h^n, v_h \right\rangle + \kappa_1 \left\langle \Delta \bar{u}_h^n, \Delta v_h \right\rangle + \kappa_2 \left\langle \nabla \bar{u}_h^n, \nabla v_h \right\rangle = \left\langle F(\bar{u}_h^n), v_h \right\rangle + \left\langle R_\tau, v_h \right\rangle, \tag{4.12}$$

where there exists a positive constant C such that $|R_\tau| \le C\tau^2$ and

$$\left\langle \bar{\delta}_t U_h^n, v_h \right\rangle + \kappa_1 \left\langle \Delta \overline{U}_h^n, \Delta v_h \right\rangle + \kappa_2 \left\langle \nabla \overline{U}_h^n, \nabla v_h \right\rangle = \left\langle F(\overline{U}_h^n), v_h \right\rangle. \tag{4.13}$$

Subtracting Eq. (4.13) from Eq. (4.12) results in

$$\left\langle \bar{\delta}_t \overline{\mathfrak{L}}_h^n, v_h \right\rangle + \kappa_1 \left\langle \Delta \overline{\mathfrak{L}}_h^n, \Delta v_h \right\rangle + \kappa_2 \left\langle \nabla \overline{\mathfrak{L}}_h^n, \nabla v_h \right\rangle = \left\langle F(\bar{u}_h^n) - F(\bar{U}_h^n), v_h \right\rangle + \left\langle R_\tau, v_h \right\rangle, \tag{4.14}$$

where

$$\overline{\mathfrak{E}}_h^n = \overline{u}_h^n - \overline{U}_h^n, \tag{4.15}$$

The assumption $v_h = \overline{\mathfrak{E}}_h^n$ yields

$$\frac{1}{2\tau}\left[\left\|\mathfrak{E}_h^n\right\|_{L^2}^2 - \left\|\mathfrak{E}_h^{n-1}\right\|_{L^2}^2\right] + \frac{1}{2}\left[\kappa_1\left\|\Delta\overline{\mathfrak{E}}_H^n\right\|_{L^2}^2 + \kappa_1\left\|\Delta\overline{\mathfrak{E}}_h^{n-1}\right\|_{L^2}^2 + \kappa_2\left\|\nabla\overline{\mathfrak{E}}_h^n\right\|_{L^2}^2 + \kappa_2\left\|\nabla\overline{\mathfrak{E}}_h^{n-1}\right\|_{L^2}^2\right]$$
$$\leq L\left\|\overline{\mathfrak{E}}_h^n\right\|_{L^2}\left\|\overline{\mathfrak{E}}_h^n\right\|_{L^2} + \|R_\tau\|_{L^2}\left\|\overline{\mathfrak{E}}_h^n\right\|_{L^2}. \tag{4.16}$$

Changing index n to i and then summing i from 1 to n result in

$$\frac{1}{2\tau}\sum_{i=1}^{n}\left[\left\|\mathfrak{E}_h^i\right\|_{L^2}^2 - \left\|\mathfrak{E}_h^{i-1}\right\|_{L^2}^2\right] + \frac{1}{2}\sum_{i=1}^{n}\left[\kappa_1\left\|\Delta\overline{\mathfrak{E}}_h^i\right\|_{L^2}^2 + \kappa_1\left\|\Delta\overline{\mathfrak{E}}_h^{i-1}\right\|_{L^2}^2\right]$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\left[\kappa_2\left\|\nabla\overline{\mathfrak{E}}_h^i\right\|_{L^2}^2 + \kappa_2\left\|\nabla\overline{\mathfrak{E}}_h^{i-1}\right\|_{L^2}^2\right]$$
$$\leq \frac{3L}{2}\sum_{i=1}^{n}\left\|\overline{\mathfrak{E}}_h^i\right\|_{L^2}^2 + \frac{1}{2L}\sum_{i=1}^{n}\|R_\tau\|_{L^2}^2. \tag{4.17}$$

Simplifying the above relations gives

$$\left[\left\|\mathfrak{E}_h^n\right\|_{L^2}^2 - \left\|\mathfrak{E}_h^0\right\|_{L^2}^2\right] + \tau\sum_{i=1}^{n}\left[\kappa_1\left\|\Delta\overline{\mathfrak{E}}_h^i\right\|_{L^2}^2 + \kappa_1\left\|\Delta\overline{\mathfrak{E}}_h^{i-1}\right\|_{L^2}^2\right]$$
$$+ \tau\sum_{i=1}^{n}\left[\kappa_2\left\|\nabla\overline{\mathfrak{E}}_h^i\right\|_{L^2}^2 + \kappa_2\left\|\nabla\overline{\mathfrak{E}}_h^{i-1}\right\|_{L^2}^2\right]$$
$$\leq 3\tau L\sum_{i=1}^{n}\left\|\overline{\mathfrak{E}}_h^i\right\|_{L^2}^2 + \frac{\tau}{L}\sum_{i=1}^{n}\|R_\tau\|_{L^2}^2. \tag{4.18}$$

Since $\overline{\mathfrak{E}}_h^0 = 0$, we have

$$\left\|\mathfrak{E}_h^n\right\|_{L^2}^2 \leq \frac{TC\tau^4}{L} + 3\tau L\sum_{i=1}^{n}\left\|\mathfrak{E}_h^i\right\|_{L^2}^2. \tag{4.19}$$

Employing Proposition 2.1 arrives at

$$\left\|\mathfrak{E}_h^n\right\|_{L^2(\Omega)}^2 \leq \frac{TC\tau^4}{L}\exp\left(\sum_{i=0}^{n}3\tau L\right) \leq \frac{TC\tau^4}{L}\exp(3TL) = C_+\tau^4, \tag{4.20}$$

thus

$$\left\|\mathfrak{E}_h^n\right\|_{L^2(\Omega)} \leq C_+\tau^2, \tag{4.21}$$

which completes the proof. ■

5. Existence and uniqueness of solution of the Galerkin weak form

Theorem 5.1. *Suppose U_H^{n-1} is known. Then the solution of (4.3) exists.*

Proof. Relation (4.3) is equivalent to

$$\left\langle\frac{U_h^n - U_h^{n-1}}{\tau}, v_h\right\rangle + \kappa_1\left\langle\frac{\Delta U_h^n + \Delta U_h^{n-1}}{2}, \Delta v_h\right\rangle + \kappa_2\left\langle\frac{\nabla U_h^n + \nabla U_h^{n-1}}{2}, \nabla v_h\right\rangle = \left\langle F(\overline{U}_h^n), v_h\right\rangle, \quad \forall v_h \in V_h,$$

or

$$\left\langle U_h^n, v_h\right\rangle + \frac{\tau\kappa_1}{2}\left\langle\Delta U_h^n, \Delta v_h\right\rangle + \frac{\tau\kappa_2}{2}\left\langle\nabla U_h^n, \nabla v_h\right\rangle = \left\langle U_h^{n-1}, v_h\right\rangle$$
$$- \frac{\tau\kappa_1}{2}\left\langle\Delta U_h^{n-1}, \Delta v_h\right\rangle - \frac{\tau\kappa_2}{2}\left\langle\nabla U_h^{n-1}, \nabla v_h\right\rangle + \tau\left\langle F(\overline{U}_h^n), v_h\right\rangle, \qquad \forall v_h \in V_h. \tag{5.1}$$

We define the function

$$\mathcal{G}(\omega) = \langle \omega, v_h \rangle + \frac{\tau\kappa_1}{2} \langle \Delta\omega, \Delta v_h \rangle + \frac{\tau\kappa_2}{2} \langle \nabla\omega, \nabla v_h \rangle - \left\langle U_h^{n-1}, v_h \right\rangle + \frac{\tau\kappa_1}{2} \left\langle \Delta U_h^{n-1}, \Delta v_h \right\rangle + \frac{\tau\kappa_2}{2} \left\langle \nabla U_h^{n-1}, \nabla v_h \right\rangle - \tau \langle F(\omega, v_h) \rangle, \quad \forall v_h \in V_h. \tag{5.2}$$

Thus, we will have

$$\langle \mathcal{G}(\omega_1) - \mathcal{G}(\omega_2), v_h \rangle = \langle \omega_1 - \omega_2, v_h \rangle + \frac{\tau\kappa_1}{2} \langle \Delta\omega_1 - \Delta\omega_2, \Delta v_h \rangle + \frac{\tau\kappa_2}{2} \langle \nabla\omega_1 - \nabla\omega_2, \nabla v_h \rangle - \tau \langle F(\omega_1) - F(\omega_2), v_h \rangle, \quad \forall v_h \in V_h. \tag{5.3}$$

In the other hand, let

$$|F(v_1) - F(v_2)| \leq L |v_1 - v_2| .$$

The Cauchy–Schwarz inequality [52] results in

$$\begin{aligned}
\langle \mathcal{G}(\omega_1) - \mathcal{G}(\omega_2), v_h \rangle &\leq \|\omega_1 - \omega_2\|_{L^2} \|v_h\|_{L^2} + \left(\frac{\tau\kappa_1}{2}\right) \|\Delta\omega_1 - \Delta\omega_2\|_{L^2} \|\Delta v_h\|_{L^2} \\
&+ \left(\frac{\tau\kappa_2}{2}\right) \|\nabla\omega_1 - \nabla\omega_2\|_{L^2} \|\nabla v_h\|_{L^2} + L\tau \|\omega_1 - \omega_2\|_{L^2} \|v_h\|_{L^2} \\
&\leq \left(\|\omega_1 - \omega_2\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta\omega_1 - \Delta\omega_2\|_{L^2} + \sqrt{\frac{\tau\kappa_2}{2}} \|\nabla\omega_1 - \nabla\omega_2\|_{L^2} \right) \\
&\times \left(\|v_h\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta v_h\|_{L^2} + \sqrt{\frac{\tau\kappa_2}{2}} \|\nabla v_h\|_{L^2} \right) + L\tau \|\omega_1 - \omega_2\|_{L^2} \|v_h\|_{L^2} \\
&\leq \left(\|\omega_1 - \omega_2\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta\omega_1 - \Delta\omega_2\|_{L^2} + \sqrt{\frac{\tau\kappa_2}{2}} \|\nabla\omega_1 - \nabla\omega_2\|_{L^2} \right) \\
&\times \left((L\tau) \|v_h\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta v_h\|_{L^2} + \sqrt{1 + \frac{\tau\kappa_2}{2}} \|\nabla v_h\|_{L^2} \right) \\
&\leq \left((1 + L\tau) \|\omega_1 - \omega_2\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta\omega_1 - \Delta\omega_2\|_{L^2} + \sqrt{\frac{\tau\kappa_2}{2}} \|\nabla\omega_1 - \nabla\omega_2\|_{L^2} \right) \\
&\times \left((L\tau) \|v_h\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta v_h\|_{L^2} + \sqrt{\frac{\tau\kappa_2}{2}} \|\nabla v_h\|_{L^2} \right),
\end{aligned} \tag{5.4}$$

or equivalently

$$\langle \mathcal{G}(\omega_1) - \mathcal{G}(\omega_2), v_H \rangle \leq \|\omega_1 - \omega_2\|_* \|v_h\|_*, \tag{5.5}$$

where

$$\|\varphi\|_* = (1 + L\tau) \|\varphi\|_{L^2} + \sqrt{\frac{\tau\kappa_1}{2}} \|\Delta\varphi\|_{L^2} + \sqrt{\frac{\tau\kappa_2}{2}} \|\nabla\varphi\|_{L^2}. \tag{5.6}$$

On the other hand, we have

$$\begin{aligned}
\langle \mathcal{G}(\omega), \omega \rangle &= \|\omega\|_{L^2}^2 + \left(\frac{\tau\kappa_1}{2}\right) \|\Delta\omega\|_{L^2}^2 + \left(\frac{\tau\kappa_2}{2}\right) \|\nabla\omega\|_{L^2}^2 \\
&- \left\langle U_H^{n-1}, \omega \right\rangle + \left(\frac{\tau\kappa_1}{2}\right) \left\langle \Delta U_H^{n-1}, \Delta\omega \right\rangle + \left(\frac{\tau\kappa_2}{2}\right) \left\langle \nabla U_H^{n-1}, \nabla\omega \right\rangle + \tau \langle f(\omega), \omega \rangle . \\
&\geq \frac{\tau}{2} \left[\|\omega\|_{L^2}^2 + \left(\frac{\tau\kappa_1}{2}\right) \|\Delta\omega\|_{L^2}^2 + \left(\frac{\tau\kappa_2}{2}\right) \|\nabla\omega\|_{L^2}^2 \right] \\
&- \frac{\tau}{2} \left[\left\| U_H^{n-1} \right\|_{L^2}^2 + \left| \frac{\tau\kappa_1}{2} \right| \left\| \Delta U_H^{n-1} \right\|_{L^2}^2 + \left| \frac{\tau\kappa_2}{2} \right| \left\| \nabla U_H^{n-1} \right\|_{L^2}^2 \right] + \frac{\tau}{2} \|f(\omega)\|_{L^2}^2 .
\end{aligned} \tag{5.7}$$

So

$$\langle \mathcal{G}(\omega), \omega \rangle \geq \frac{\tau}{2} \|\omega\|_{H^2(\Omega)}^2 - \frac{\tau}{2} \left\| U_H^{n-1} \right\|_{H^2(\Omega)}^2 + \frac{\tau}{2} \|f(\omega)\|_{L^2}^2 . \tag{5.8}$$

Putting $\|\vartheta\|_{H^2(\Omega)} = \left\| U_H^{n-1} \right\|_{H^2(\Omega)}$ yields $\langle \mathcal{G}(\omega), \omega \rangle \geq 0$. The use of the fixed point theorem yields

$$\exists \omega^* \quad : \quad F(\omega^*) = 0. \quad \blacksquare$$

Theorem 5.2. *If U_h^{n-1} is known then the solution of (4.3) is unique.*

Proof. Now, we define

$$W_h^n = U_{1h}^n - U_{2h}^n, \tag{5.9}$$

then, we can write

$$\left\langle \bar{\delta}_t W_h^n, v_h \right\rangle + \left\langle \Delta \overline{W}_h^n, \Delta v_h \right\rangle + \left\langle \nabla \overline{W}_h^n, \nabla v_h \right\rangle = \left\langle f(\bar{U}_{1h}^n) - f(\bar{U}_{2h}^n), v_h \right\rangle, \qquad \forall v_h \in J_h. \tag{5.10}$$

Setting $v_h = W_h^n$ yields

$$\begin{aligned} \frac{1}{2\tau}\left[\left\|W_h^n\right\|_{L^2}^2 - \left\|W_h^{n-1}\right\|_{L^2}^2\right] + \frac{1}{2}\left[\kappa_1 \left\|\Delta W_H^n\right\|_{L^2}^2 + \kappa_1 \left\|\Delta W_H^{n-1}\right\|_{L^2}^2 + \kappa_2 \left\|\nabla W_H^n\right\|_{L^2}^2 + \kappa_2 \left\|\nabla W_H^{n-1}\right\|_{L^2}^2\right] \\ \leq L\left\|\bar{W}_h^n\right\|_{L^2}\left\|\bar{W}_h^n\right\|_{L^2}. \end{aligned} \tag{5.11}$$

Summing i from 1 to n results in

$$\begin{aligned} \frac{1}{2\tau}\sum_{i=1}^{n}\left[\left\|W_h^i\right\|_{L^2}^2 - \left\|W_h^{i-1}\right\|_{L^2}^2\right] + \frac{1}{2}\sum_{i=1}^{n}\left[\kappa_1 \left\|\Delta W_H^i\right\|_{L^2}^2 + \kappa_1 \left\|\Delta W_H^{i-1}\right\|_{L^2}^2\right] \\ + \frac{1}{2}\sum_{i=1}^{n}\left[\kappa_2 \left\|\nabla W_H^i\right\|_{L^2}^2 + \kappa_2 \left\|\nabla W_H^{i-1}\right\|_{L^2}^2\right] \leq L\sum_{i=1}^{n}\left\|\bar{W}_h^i\right\|_{L^2}^2. \end{aligned} \tag{5.12}$$

Simplifying the above relations outputs

$$\begin{aligned} \left[\left\|W_h^n\right\|_{L^2}^2 - \left\|W_h^0\right\|_{L^2}^2\right] + \tau\sum_{i=1}^{n}\left[\kappa_1 \left\|\Delta W_H^i\right\|_{L^2}^2 + \kappa_1 \left\|\Delta W_H^{i-1}\right\|_{L^2}^2\right] \\ + \tau\sum_{i=1}^{n}\left[\kappa_2 \left\|\nabla W_H^i\right\|_{L^2}^2 + \kappa_2 \left\|\nabla W_H^{i-1}\right\|_{L^2}^2\right] \leq 2L\tau\sum_{i=1}^{n}\left\|\bar{W}_h^i\right\|_{L^2}^2, \end{aligned} \tag{5.13}$$

or equivalently

$$\left\|W_h^n\right\|_{L^2}^2 \leq \left\|W_h^0\right\|_{L^2}^2 + 2L\tau\sum_{i=1}^{n}\left\|W_h^i\right\|_{L^2}^2. \tag{5.14}$$

Applying Proposition 2.1 gives

$$\left\|W_h^n\right\|_{L^2(\Omega)}^2 \leq \left\|W_h^0\right\|_{L^2(\Omega)}^2 \exp\left(\sum_{i=0}^{n} 2\tau \mathcal{L} C_\Omega\right), \tag{5.15}$$

and because of $W_h^0 = 0$ we can write

$$\left\|W_h^n\right\|_{L^2}^2 = 0 \qquad \Rightarrow \qquad W_h^n = 0, \tag{5.16}$$

which completes the proof. ■

6. Convergence analysis of full-discrete scheme

Here, we describe some preliminaries which will be used in the current section. We define

$$V_h = span\left\{\varphi_1, \varphi_2, \ldots, \varphi_d\right\}, \tag{6.1}$$

and consider the Ritz-projection

$$\Upsilon_r^d : H_0^2(\Omega) \to V_h, \tag{6.2}$$

such that for $v \in V_h$

$$\left(\nabla \Upsilon_r^d v, \nabla u_r\right) = \left(\nabla v, \nabla u_r\right), \qquad \forall\ u_r \in V_h. \tag{6.3}$$

Lemma 6.1 ([32]). *If $v(\mathbf{x}) \in C^{m+1}(\Omega)$ then there exist constants C_ν such that*

$$\left\|D^\nu v - D^\nu \Upsilon_r^d v\right\|_{L^2(\Omega)} \leq \mathbf{C}_\nu \delta^{m+1-|\nu|}|v|_{m,1}, \qquad |\nu| \leq 2.$$

Theorem 6.2. *Let u^n and U_h^n be corresponding solutions for schemes (4.2) and (4.3), respectively. Then*

$$\left\|u^n - U_h^n\right\|_{H^2(\Omega)} \leq C(\tau^2 + \delta^{m+1}). \tag{6.4}$$

Proof. We set

$$u(t_n) - U_h^n = (u(t_n) - u_h(t_n)) + \left(u_h(t_n) - U_h^n\right), \tag{6.5}$$

$$\rho_h^n = U_h^n - u_h(t_n) = U_h^n - u_h^n. \tag{6.6}$$

Thus the main weak form can be changed to

$$\begin{aligned}\left\langle\bar{\delta}_t\rho_h^n, v_h\right\rangle + \kappa_1\left\langle\Delta\bar{\rho}_h^n, \Delta v_h\right\rangle + \kappa_2\left\langle\nabla\bar{\rho}_h^n, \nabla v_h\right\rangle \\ = \left\langle\vartheta_h^n, v_h\right\rangle + \left\langle\Delta\vartheta_h^n, \Delta v_h\right\rangle + \left\langle\nabla\vartheta_h^n, \nabla v_h\right\rangle + \zeta_{1h}(v_h),\end{aligned} \tag{6.7}$$

where

$$\zeta_{1h}(v_h) = \left\langle F(\overline{U}_h^n), v_h\right\rangle - \left\langle F(\overline{u}_h^n), v_h\right\rangle, \tag{6.8}$$

and

$$\vartheta_h^n = \overline{u_h^n} - \bar{\delta}_t u_h^n. \tag{6.9}$$

Assuming $v_h = \overline{\rho}_h^n$ in Eq. (6.7) results in

$$\left\langle\bar{\delta}_t\rho_h^n, \overline{\rho}_h^n\right\rangle + \kappa_1\left\langle\Delta\overline{\rho}_h^n, \Delta\overline{\rho}_h^n\right\rangle + \kappa_2\left\langle\nabla\overline{\rho}_h^n, \nabla\overline{\rho}_h^n\right\rangle = \left\langle\vartheta_h^n, \overline{\rho}_h^n\right\rangle + \xi_{1h}(\overline{\rho}_h^n). \tag{6.10}$$

Summing (6.10) for i from 1 to n gives

$$\frac{1}{2\tau}\left\|\rho_h^n\right\|_{L^2}^2 + \kappa_1\sum_{i=1}^{n}\left\|\Delta\overline{\rho}_h^i\right\|_{L^2}^2 + \kappa_2\sum_{i=1}^{n}\left\|\nabla\overline{\rho}_h^i\right\|_{L^2}^2 = \sum_{i=1}^{n}\left\langle\vartheta_H^i, \overline{\rho}_h^i\right\rangle + \sum_{i=1}^{n}\xi_{1h}(\overline{\rho}_h^i), \tag{6.11}$$

or

$$\left\|\rho_h^n\right\|_{L^2}^2 + 2\tau\kappa_1\left\|\Delta\overline{\rho}_h^n\right\|_{L^2}^2 + 2\tau\kappa_2\left\|\nabla\overline{\rho}_h^n\right\|_{L^2}^2 \leq 2\tau\sum_{i=1}^{n}\left\langle\vartheta_H^i, \overline{\rho}_h^i\right\rangle + 2\tau\sum_{i=1}^{n}\xi_{1h}(\overline{\rho}_h^i). \tag{6.12}$$

Now

$$\vartheta_h^i = \overline{u_{ht}^i} - \bar{\delta}_t u_h^i = \overline{u_{ht}^i} - \frac{1}{\tau}\int_{t_{i-1}}^{t_i} u_{ht}(s)ds = \frac{1}{\tau}\int_{t_{i-1}}^{t_i}\lambda_i(t)u_{ht}(s)ds, \tag{6.13}$$

where

$$\lambda_i(t) = (t_i - t)(t - t_{i-1}). \tag{6.14}$$

On the other hand, we can write

$$\begin{aligned}\left\|\vartheta_h^i\right\|_{L^2} &\leq \frac{1}{2\tau}\left(\int_{t_{i-1}}^{t_i}(t_i - t)^2 dt\right)^{\frac{1}{2}}\left(\int_{t_{i-1}}^{t_i}(t - t_{i-1})^2\left\|u_{httt}\right\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\leq C\tau^{\frac{1}{2}}\left(\int_{t_{i-1}}^{t_i}(t - t_{i-1})^2\left\|u_{httt}\right\|_{L^2}^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{6.15}$$

So, we can get

$$\begin{aligned}\tau\sum_{i=1}^{n}\left\|\vartheta_h^i\right\|_{L^2} &\leq C\tau^2\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t - t_{i-1})^2\left\|u_{httt}\right\|_{L^2}^2 dt \\ &\leq C\tau^4\int_{t_0}^{t_n}\left\|u_{httt}\right\|_{L^2}^2 dt \leq C\tau^4\int_{t_0}^{T}\left\|u_{httt}\right\|_{L^2}^2 dt \leq C\tau^4.\end{aligned}$$

Taking the Cauchy–Schwarz inequality, Eq. (6.12) becomes

$$\left\|\rho_h^n\right\|_{L^2}^2 + 2\tau\kappa_1\left\|\Delta\overline{\rho}_h^n\right\|_{L^2}^2 + 2\tau\kappa_2\left\|\nabla\overline{\rho}_h^n\right\|_{L^2}^2 \leq C\tau^4 + \tau\,(1 + 2L)\sum_{i=1}^{n}\left\|\overline{\rho}_h^i\right\|_{L^2}^2. \tag{6.16}$$

From the above equations and by employing Proposition 2.1 we achieve

$$\begin{aligned}\left\|\rho_h^n\right\|_{L^2}^2+2\tau\kappa_1\left\|\Delta\overline{\rho}_h^n\right\|_{L^2}^2+2\tau\kappa_2\left\|\nabla\overline{\rho}_h^n\right\|_{L^2}^2 &\le C\tau^4+\tau\left(1+2L\right)\sum_{i=1}^{n}\left\|\overline{\rho}_h^i\right\|_{L^2}^2\\ &\le C\tau^4\exp\left(\tau\left(1+2L\right)\right)\le C^*\tau^4.\end{aligned} \tag{6.17}$$

Finally, we obtain

$$\left\|\rho_h^n\right\|_{H^2(\Omega)}\le \mathcal{C}\tau^2. \tag{6.18}$$

According to Eq. (6.5) we have

$$\left\|u^n-U_h^n\right\|_{L^2(\Omega)}\le\left\|u^n-u_h^n\right\|_{L^2(\Omega)}+\left\|u_h^n-U_h^n\right\|_{L^2(\Omega)}. \tag{6.19}$$

The inequality (6.18) and

$$\left\|u^n-u_h^n\right\|_{L^2(\Omega)}\le C_1N^{1-m},\qquad \left\|u_h^n-U_h^n\right\|_{L^2(\Omega)}\le C_2\tau^2, \tag{6.20}$$

complete the proof. ■

7. Numerical analysis

In the current section, we investigate the convergence and stability of the developed algorithm for three tests. Also, the computational rate is calculated by

$$\mathrm{C}_2\text{−order}=\frac{\log\left(\frac{\|L_\infty(\tau,h_1)\|}{\|L_\infty(\tau,h_2)\|}\right)}{\log\left(\frac{h_1}{h_2}\right)},$$

where $L_\infty(\tau,h)$ presents the maximum error which corresponds to a grid with mesh size τ and h.

7.1. Test problem 1

At first, we consider an example with an analytical solution [3,4], namely the equation

$$\frac{\partial u}{\partial t}+\kappa_1u_{xxxx}-\kappa_2u_{xx}=u-u^3,\qquad x\in[a,b], \tag{7.1}$$

with $\kappa_1=\kappa_2=1$ and the boundary conditions

$$\begin{cases} u(a,t)=\exp(-t)\sin(\pi a), & u(b,t)=\exp(-t)\sin(\pi b),\\ u_{xx}(a,t)=-\pi^2\exp(-t)\sin(\pi a), & u_{xx}(b,t)=-\pi^2\exp(-t)\sin(\pi b)\end{cases}$$

and the initial condition

$$u(x,0)=\sin(\pi x),\qquad (x,y)\in[a,b],$$

and also

$$f(x,t)=\exp(-3t)\sin(\pi x)^3-2\exp(-t)\sin(\pi x)+\pi^2\exp(-t)\sin(\pi x)+\pi^4\exp(-t)\sin(\pi x).$$

Then, the analytical solution of this problem will be

$$u(x,t)=\exp(-t)\sin(\pi x).$$

We solve this example by using the proposed method to check the stability and convergence of the developed technique. In this example, we assume that $m=2$. Table 1 shows the errors and computational orders obtained for Test Problem 1. In Table 1 the time step is fixed and then the mesh size is changed. The results confirm the convergence order of the proposed method in the space direction.

Table 2 shows the error and computational orders obtained for Test Problem 1. In Table 2 the mesh size is fixed and then the time step is changed. The results present that the computational order is closed to the theoretical order i.e. $O(\tau^2)$. Fig. 1 demonstrates the numerical solutions at various final times (left panel) and the corresponding absolute error (right panel) for Test problem 1.

Table 1
Errors and computational orders obtained for Test Problem 1.

	h	$\tau = 10^{-2}$		$\tau = 10^{-3}$	
		L_∞	C_2-order	L_∞	C_2-order
	1/10	7.7195×10^{-2}	–	2.1542×10^{-2}	–
	1/20	2.1536×10^{-2}	1.8417	5.8138×10^{-3}	1.8896
	1/40	5.8078×10^{-3}	1.8907	1.4625×10^{-3}	1.9911
$T = 1$	1/80	1.4567×10^{-3}	1.9953	3.6573×10^{-4}	1.9996
	1/160	3.5992×10^{-4}	2.0169	9.0978×10^{-5}	2.0072
	1/320	8.5181×10^{-5}	2.0791	2.2256×10^{-5}	2.0313
	1/640	1.6461×10^{-5}	2.3715	5.0729×10^{-6}	2.1333
	h	$\tau = 10^{-2}$		$\tau = 10^{-3}$	
		L_∞	C_2-order	L_∞	C_2-order
	1/10	2.8445×10^{-2}	–	2.8442×10^{-3}	–
	1/20	7.9425×10^{-3}	1.8405	7.9380×10^{-4}	1.8412
	1/40	2.1437×10^{-3}	1.8895	2.1386×10^{-4}	1.8921
$T = 2$	1/80	5.3904×10^{-4}	1.9917	5.3377×10^{-5}	2.0042
	1/160	1.3454×10^{-4}	2.0023	1.2924×10^{-5}	2.0461
	1/320	3.3208×10^{-5}	2.0184	2.7903×10^{-6}	2.2116
	1/640	7.8616×10^{-6}	2.0786	2.5545×10^{-7}	3.4493

Table 2
Errors and computational orders obtained for Test Problem 1.

τ	$h = 10^{-2}$		$h = 0.5 \times 10^{-2}$		CPU time(s)
	L_∞	C_2-order	L_∞	C_2-order	
1/10	7.7141×10^{-2}	–	1.3473×10^{-3}	–	4
1/20	2.1512×10^{-2}	1.8424	4.8692×10^{-4}	1.4683	9
1/40	5.7983×10^{-3}	1.8914	1.2923×10^{-4}	1.9137	21
1/80	1.4551×10^{-3}	1.9946	3.2750×10^{-5}	1.9804	47
1/160	3.6232×10^{-4}	2.0057	8.1973×10^{-6}	1.9983	89
1/320	8.9593×10^{-5}	2.0158	2.0410×10^{-6}	2.0059	112
1/640	2.1888×10^{-5}	2.0332	5.0524×10^{-7}	2.0142	178

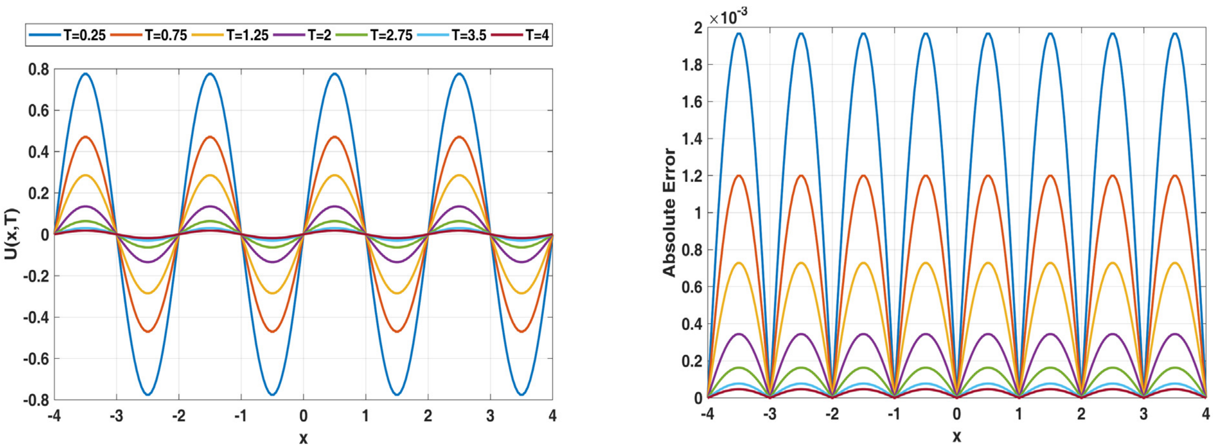

Fig. 1. Numerical solutions and graph of absolute errors at various final times for Test Problem 1.

7.2. Test problem 2

We study the following example without an analytical solution [3,4]

$$\frac{\partial u}{\partial t} + \kappa_1 u_{xxxx} - \kappa_2 u_{xx} = u - u^3, \qquad x \in [a, b], \tag{7.2}$$

with boundary conditions

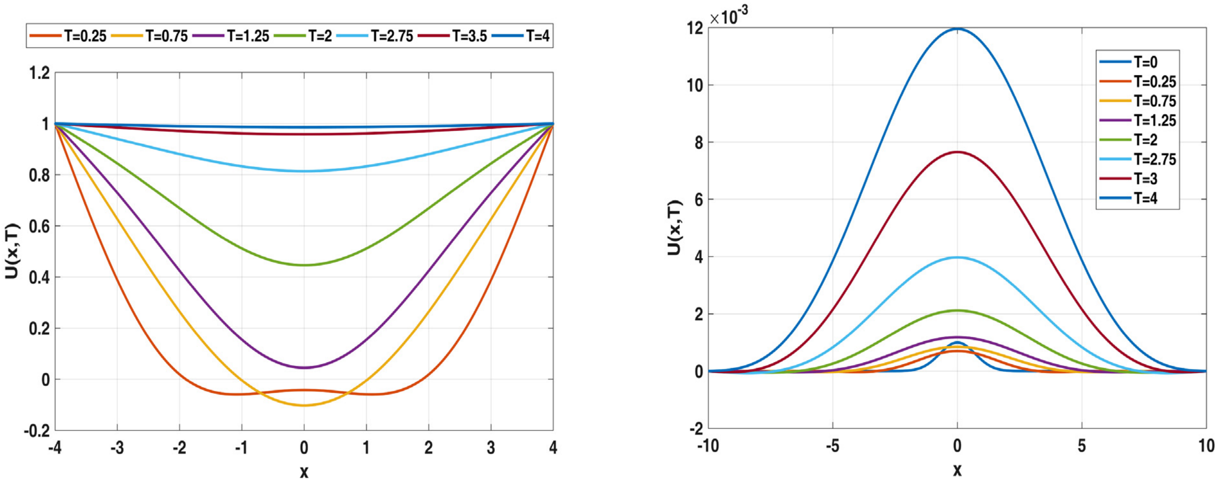

Fig. 2. Numerical solutions at various final times for Test Problem 2.

Table 3
Errors and computational orders obtained for Test problem 3.

	h	$h = 1/200$		$h = 1/400$	
		L_∞	C_2-order	L_∞	C_2-order
$T = 1$	1/10	1.0548×10^{-2}	–	1.3453×10^{-3}	–
	1/20	2.7208×10^{-3}	1.9548	3.5223×10^{-4}	1.9333
	1/40	6.9127×10^{-4}	1.9767	8.9778×10^{-5}	1.9721
	1/80	1.7424×10^{-4}	1.9881	2.2646×10^{-5}	1.9871
	1/160	4.3742×10^{-5}	1.9940	5.6861×10^{-6}	1.9938
	1/320	1.0958×10^{-5}	1.9970	1.4245×10^{-6}	1.9969
$T = 4$	1/10	2.5354×10^{-3}	–	1.8104×10^{-4}	–
	1/20	6.7445×10^{-4}	1.9104	4.0113×10^{-5}	2.1742
	1/40	1.7248×10^{-4}	1.9673	1.0020×10^{-5}	2.0012
	1/80	4.3540×10^{-5}	1.9860	2.5044×10^{-6}	2.0003
	1/160	1.0934×10^{-5}	1.9935	6.2606×10^{-7}	2.0001
	1/320	2.7395×10^{-6}	1.9969	1.5651×10^{-7}	2.0000
$T = 8$	1/10	3.1391×10^{-5}	–	6.4113×10^{-6}	–
	1/20	7.8471×10^{-6}	2.0001	1.6027×10^{-6}	2.0001
	1/40	1.9616×10^{-6}	2.0001	4.0067×10^{-7}	2.0000
	1/80	4.9039×10^{-7}	2.0000	1.0017×10^{-7}	2.0000
	1/160	1.2260×10^{-7}	2.0000	2.5042×10^{-8}	2.0000
	1/320	3.0649×10^{-8}	2.0000	6.2605×10^{-9}	2.0000

$$\begin{cases} u(a,t)=0, \quad u(b,t)=0, \qquad \text{or} \qquad u(a,t)=1, \quad u(b,t)=1, \\ u_{xx}(a,t)=0, \quad u_{xx}(b,t)=0, \end{cases}$$

and the initial condition

$$u(x,0) = 10^{-3}\exp(-x^2).$$

We solve this example by using the proposed method to check the stability and convergence of the developed technique. In this example, there is not any analytical solution. We consider this test problem to find some approximate solutions. Here, we consider two different boundary conditions i.e. $u(a,t) = 0$ and $u(b,t) = 0$ as well as $u(a,t) = 1$ and $u(b,t) = 1$ where the initial condition is based on the Gaussian pulse. The left hand side of Fig. 2 is based on $u(a,t) = 1$ and $u(b,t) = 1$ whereas the right hand side of Fig. 2 is obtained by using $u(a,t) = 0$ and $u(b,t) = 0$. From, Fig. 2 it is clear that different boundary conditions affect the final solution.

Table 4
Errors and computational orders obtained for Test problem 3.

	h	$\tau = 10^{-2}$		$\tau = 10^{-3}$	
		L_∞	C_2-order	L_∞	C_2-order
$T = 1$	1/10	9.0404×10^{-3}	–	9.0398×10^{-4}	–
	1/20	2.2833×10^{-3}	1.9853	2.2827×10^{-4}	1.9855
	1/40	5.7223×10^{-4}	1.9964	5.7164×10^{-5}	1.9976
	1/80	1.4310×10^{-4}	1.9996	1.4251×10^{-5}	2.0046
	1/160	3.5732×10^{-5}	2.0018	3.5142×10^{-6}	2.0198
	1/320	8.8841×10^{-6}	2.0079	8.2948×10^{-7}	2.0829
	1/640	2.1718×10^{-6}	2.0323	1.5826×10^{-7}	2.3899
	h	$\tau = 10^{-2}$		$\tau = 10^{-3}$	
		L_∞	C_2-order	L_∞	C_2-order
$T = 2$	1/10	3.5339×10^{-4}	–	2.1956×10^{-5}	–
	1/20	9.2186×10^{-5}	1.9386	5.9297×10^{-6}	1.8886
	1/40	2.3289×10^{-5}	1.9849	1.4924×10^{-6}	1.9904
	1/80	5.8372×10^{-6}	1.9963	3.7368×10^{-7}	1.9977
	1/160	1.4599×10^{-6}	1.9994	9.3426×10^{-8}	1.9999
	1/320	3.6471×10^{-7}	2.0001	2.3326×10^{-8}	2.0019
	1/640	9.0846×10^{-8}	2.0052	5.7981×10^{-9}	2.0083

7.3. Test problem 3

For the last test problem, we consider the two-dimensional model [3,4]

$$\begin{cases} \dfrac{\partial u}{\partial t} + \kappa_1 \Delta^2 u - \kappa_2 \Delta u = u - u^3, & \text{in } \Omega, \\ u = \exp(-t)\sin(2\pi x)\sin(2\pi y), \quad \Delta u = -8\pi^2 \exp(-t)\sin(2\pi x)\sin(2\pi y), & \text{on } \partial\Omega, \\ u(\mathbf{x}, 0) = \sin(2\pi x)\sin(2\pi y), & \text{in } \Omega, \end{cases} \tag{7.3}$$

where the exact solution is

$$u = \exp(-t)\sin(2\pi x)\sin(2\pi y).$$

In this problem $\Omega = [-1, 1] \times [-1, 1]$. We obtain the numerical solutions based upon the developed technique.

Tables 3 and 4 present the errors and computational orders obtained for Test Problem 3. In Table 3, we fix the time step and consider the effect of the mesh size on the computational error. In Table 4, the time step is assumed fixed and the mesh size is decreasing. All obtained results verify the expected theoretical orders in the temporal and spatial directions, respectively.

8. Conclusion

In this paper, we developed a new numerical procedure to solve the nonlinear extended Fisher–Kolmogorov equation. In the first step, the time variable was discretized by a second-order difference formula. Then, by using the energy method we proved the order of convergence and the unconditional stability. Furthermore, the existence and uniqueness of the solution of the proposed technique were studied in detail. At the second step, the error estimate of the full-discrete formulation, based upon the interpolating element free Galerkin approach (based on the interpolation MLS approximation) was investigated. The achieved convergence order of fully discrete plan is $O(\tau^2 + \delta^{m+1})$ where τ, δ and m denote the time step, the radius of the weight function, and smoothness of the exact solution of the main problem, respectively. The used three specific test problems and the numerical results confirm the theoretical analysis addressed in this work and the capability of the new algorithm.

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