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Galerkin proper orthogonal decomposition-reduced order method (POD-ROM) for solving generalized Swift-Hohenberg equation

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Abstract

Purpose – The current paper aims to develop a reduced order discontinuous Galerkin method for solving the generalized Swift–Hohenberg equation with application in biological science and mechanical engineering. The generalized Swift–Hohenberg equation is a fourth-order PDE; thus, this paper uses the local discontinuous Galerkin (LDG) method for it.

Design/methodology/approach – At first, the spatial direction has been discretized by the LDG technique, as this process results in a nonlinear system of equations based on the time variable. Thus, to achieve more accurate outcomes, this paper uses an exponential time differencing scheme for solving the obtained system of ordinary differential equations. Finally, to decrease the used CPU time, this study combines the proper orthogonal decomposition approach with the LDG method and obtains a reduced order LDG method. The circular and rectangular computational domains have been selected to solve the generalized Swift–Hohenberg equation. Furthermore, the energy stability for the semi-discrete LDG scheme has been discussed.

Findings – The results show that the new numerical procedure has not only suitable and acceptable accuracy but also less computational cost compared to the local DG without the proper orthogonal decomposition (POD) approach.

Originality/value – The local DG technique is an efficient numerical procedure for solving models in the fluid flow. The current paper combines the POD approach and the local LDG technique to solve the generalized Swift–Hohenberg equation with application in the fluid mechanics. In the new technique, the computational cost and the used CPU time of the local DG have been reduced.

Keywords Exponential time differencing (ETD) scheme, Local discontinuous Galerkin method, Swift–Hohenberg equation

Paper type Research paper

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1. Introduction

The discontinuous Galerkin (DG) method is one of the improvements of the finite element method. The DG technique has been used for solving several physical models such as computational fluid dynamics (Cockburn, 2001; Li, 2006), convection-dominated diffusion problems (Badia and Hierro, 2015; Cockburn and Shu, 1989; Demkowicz *et al.*, 2012; Ellis *et al.*, 2014), the nonlinear Hamilton-Jacobi equations (Cheng and Shu, 2007, Hu and Shu, 1998), second-order elliptic problems (Arnold *et al.*, 2002), time-dependent convection-diffusion systems (Cockburn and Shu, 1998a), nonlinear Schrödinger equations (Ai and Li, 2005; Liang *et al.*, 2015), 2D Brusselator system (Dehghan and Abbaszadeh, 2016b), multidimensional thermal radiation problems (Cui and Li, 2004), elliptic eigenvalue problems (Giani, 2015), viscous Burgers–Poisson system (Ploymaklam *et al.*, 2016) and fully coupled microscopic SNPP system (Frank *et al.*, 2015; Frank *et al.*, 2011). The main aim of (Marti *et al.*, 2017) is to propose a new elemental enrichment technique to improve the accuracy of the simulations of thermal problems containing weak discontinuities. Authors of (Karakus *et al.*, 2018) developed a high-order discontinuous Galerkin method for the solution of unsteady, incompressible, multiphase flows with level set interface formulation. A fully discrete local discontinuous Galerkin (LDG) finite element method has been proposed in (Wei *et al.*, 2013) for solving a time-fractional advection-diffusion equation. To find information for DG method the interested readers can refer to (Chou *et al.*, 2014; Cockburn *et al.*, 1990; Shu, 2014; Wang *et al.*, 2015; Zhang and Shu, 2010).

The Swift–Hohenberg equation has been introduced in Swift and Hohenberg (1977) as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\mu(u) - Dk^4 u - 2Dk^2\Delta u - D\Delta^2 u, \text{ in } \mathbf{\Omega}\times[0,T], \\[2ex] \dfrac{\partial u}{\partial \zeta} = 0, \dfrac{\partial}{\partial \zeta}\left(2Dk^2 u + D\Delta u\right) = 0, \qquad \text{on } \partial\mathbf{\Omega}\times[0,T], \\[2ex] u(\mathbf{x},0) = u_0(\mathbf{x}), \forall \mathbf{x}\in\mathbf{\Omega}, \end{cases} \tag{1.1}$$

in which $k, D, \in \mathbb{R}^+$. Equation (1.1) has some applications in Gabbrielli (2009):

- foams physics;
- cellular materials;
- crystallography;
- biology science;
- metallurgy; and
- data compression.

Mathematical model (1.1) is solved by using different approaches for instance Gomez and Nogueira (2012a) developed a new numerical procedure with the nonlinear stability property. In other hand, Gomez and Nogueira (2012a) proposed the Galerkin *B*-spline method to solve equation (1.1). Akyildiz *et al.* (2010) presented a semi-analytic approach based homotopy analysis method (HAM). Sanchez *et al.* used the finite difference method to simulate equation (1.1). Lloyd *et al.* (2008) investigated several numerical procedures for stationary spatially localized hexagon patterns. Also, Zhao *et al.* proposed the Fourier spectral procedure to analogize the Swift–Hohenberg equation. Furthermore, the interested readers can refer to (Kudryashov and Sinelshchikov, 2012; McCalla and Sandstede, 2010; Park and Park, 2014; Thiele *et al.*, 2013). Also, there are some numerical methods to solve the some equations in the biology field. For example, authors of (Dehghan *et al.*, 2011) used He's Exp-function method (EFM) to construct solitary and soliton solutions of the nonlinear evolution equation. The main aim of (Dehghan *et al.*, 2012) is to present the solution of the Rosenau–Hyman equation by using the semianalytical approaches based on the homotopy perturbation method (HPM), variational iteration method (VIM) and Adomian decomposition method (ADM).

1.1 The structure of paper

The main purpose is to find a new numerical procedure to simulate generalized Swift–Hohenberg equation based on the LDG method. We used the LDG approach for discretizing the spatial direction that leads to a nonlinear system of ordinary differential equations. Finally, we solve the obtained system using the exponential time differencing (ETD) scheme Also, we will obtain the energy stability for the semi-discrete LDG scheme. The rest of the current paper is as follows: in Section 2, the local DG method has been used to discrete the main model; in Section 3, we describe the proper orthogonal decomposition method and how to build the new bases; in Section 4, some examples have been considered to illustrate the efficiency of the proposed technique; and in Section 5, the conclusion of the paper has been proposed in this section.

2. The local discontinuous Galerkin approximation

In the current section, we present a brief mathematical introduction for the LDG method. At first, we introduce some notations. Let T_h be a regular triangulation for the computational domain in which K denotes an arbitrary triangle element and also

$$h = \max_K \{\mathrm{diam}(K)\}.$$

Let n_T be the unit normal on ∂K. For any interior triangle, two triangles K^- and K^+ are common. Now, we define

$$\zeta^{\pm}(\mathbf{x}) = \lim_{\delta \to 0^+} \zeta(\mathbf{x} - \delta \mathbf{n}_{K^{\pm}}). \tag{2.1}$$

The average and jump of ζ on each edge can be defined as

$$\{|\zeta|\} = \frac{1}{2}(\zeta^- + \zeta^+), [[\zeta]] = \zeta^- \mathbf{n}_{K^-} + \zeta^+ \mathbf{n}_{K^+}, \tag{2.2}$$

respectively.

To implement the LDG method, equation (1) must be changed as follows:

$$u_t = -\mu(u) - Dk^4 u - \nabla \cdot \mathbf{v}, \tag{2.3}$$

$$\mathbf{v} = \nabla w, \tag{2.4}$$

$$w = 2Dk^2 u + D\nabla \cdot z, \tag{2.5}$$

$$\mathbf{z} = \nabla u. \tag{2.6}$$

The LDG scheme to solve the system (2.3)-(2.6) is as follows:

Find $u, w \in V_h$ and $\mathbf{v}, \mathbf{z} \in V_h^d$ such that, for all test functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in V_h$ and $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in V_h^d$

$$\int_K \frac{\partial u}{\partial t}\varphi_1 dK = -\int_K \mu(u)\varphi_1 dK - Dk^4\int_K u\varphi_1 dK + \int_K \mathbf{v}\cdot\nabla\varphi_1 dK - \int_{\partial K}\hat{\mathbf{v}}\cdot\vec{\mathbf{n}}\varphi_1 ds, \tag{2.7}$$

$$\int_K \mathbf{v}\cdot\boldsymbol{\theta}_1 dK = -\int_K w\cdot\nabla\boldsymbol{\theta}_1 dK + \int_{\partial K}\hat{w}\cdot\vec{\mathbf{n}}\boldsymbol{\theta}_1 ds, \tag{2.8}$$

$$\int_K w\varphi_2 dK = 2Dk^2\int_K u\varphi_2 dK - D\int_K \mathbf{z}\cdot\nabla\varphi_2 dK + D\int_{\partial K}\hat{\mathbf{z}}\cdot\vec{\mathbf{n}}\varphi_2 ds, \tag{2.9}$$

$$\int_K \mathbf{z}\cdot\boldsymbol{\theta}_2 dK = -\int_K u\cdot\nabla\boldsymbol{\theta}_2 dK + \int_{\partial K}\hat{u}\cdot\vec{\mathbf{n}}\boldsymbol{\theta}_2 ds. \tag{2.10}$$

Theorem 2.1. (energy stability for the semi-discrete LDG scheme) The solution of LDG scheme (2.3)-(2.6) satisfies the energy dissipative:

$$\frac{d}{dt}\int_\Omega\left(\Psi(u) + \frac{D}{2}\left[2\left(w - 2Dk^2u\right)^2 - 2k^2\mathbf{z}\cdot\mathbf{z} + k^4u^2\right]\right)dx \leq 0. \tag{2.11}$$

Proof. We select the following test functions:

$$\varphi_1 = u_t, \qquad \varphi_2 = w_t - 2k^2u_t, \qquad \boldsymbol{\theta}_2 = \mathbf{z}_t. \tag{2.12}$$

Substituting the above test functions in equations (2.7)-(2.10), we have:

$$\int_K (u_t)^2 dK = -\int_K \mu(u)u_t dK - Dk^4\int_K uu_t dK$$
$$+\int_K \mathbf{v}\cdot\nabla u_t dK - \int_{\partial K}\hat{\mathbf{v}}\cdot\vec{\mathbf{n}}u_t ds, \tag{2.13}$$

$$\int_K \left(w - 2k^2Du\right)\left(w_t - 2k^2Du_t\right)dK = -D\int_K \mathbf{z}\cdot\nabla\left(w_t - 2k^2Du_t\right)dK$$
$$+D\int_{\partial K}\hat{\mathbf{z}}\cdot\vec{\mathbf{n}}\left(w_t - 2k^2Du_t\right)ds, \tag{2.14}$$

$$D\int_K \mathbf{z}\cdot\mathbf{z}_t dK = -D\int_K u\cdot\nabla\mathbf{z}_t dK + D\int_{\partial K}\hat{u}\cdot\vec{\mathbf{n}}\,\mathbf{z}_t ds. \tag{2.15}$$

Thus, we can write:

$$\frac{d}{dt}\int_\Omega\left(\Psi(u)+\frac{D}{2}\left[\left(w-2Dk^2u\right)^2-2k^2\mathbf{z}\cdot\mathbf{z}+k^4u^2\right]\right)dx+\int_K (u_t)^2 dK$$

$$=\int_K \mathbf{v}\cdot\nabla u_t dK-\int_{\partial K}\hat{\mathbf{v}}\cdot\mathbf{n}\;u_t ds$$

$$-D^2\int_K z\cdot\nabla\left(w_t-2kD^2u_t\right)dK+D^2\int_{\partial K}(\hat{\mathbf{z}}\cdot\mathbf{n})\left(w_t-2kD^2u_t\right)ds$$

$$+D\int_K u\nabla\mathbf{z}_t dK-D\int_{\partial K}(\hat{u}\cdot n)\;\mathbf{z}_t ds.$$

Finally, summing up the above relation over K and noticing the fluxes are from the opposite sides of ∂K as well as the periodic boundary conditions, we have:

$$\frac{d}{dt}\int_\Omega\left(\Psi(u)+\frac{D}{2}\left[\left(w-2Dk^2u\right)^2-2k^2\mathbf{z}\cdot\mathbf{z}+k^4u^2\right]\right)dx+\int_K (u_t)^2 dK=0, \tag{2.16}$$

which completes the proof. □

Now, we explain the implementation of LDG method for the generalized Swift–Hohenberg equation. We rewrite equations (2.7)-(2.10) as follows:

$$\int_K \varphi_1 u_t dK=-\int_K \varphi_1\mu(u)dK-Dk^4\int_K\varphi_1 u dK+\int_K(\nabla\varphi_1)\cdot\mathbf{v}dK-\int_{\partial K}(\varphi_1)\hat{\mathbf{v}}\cdot\vec{\mathbf{n}}ds, \tag{2.17}$$

$$\int_K \boldsymbol{\theta}_1\cdot\mathbf{v}dK=-\int_K(\nabla\boldsymbol{\theta}_1)\cdot w dK+\int_{\partial K}(\boldsymbol{\theta}_1)\hat{w}\cdot\vec{\mathbf{n}}ds, \tag{2.18}$$

$$\int_K \varphi_2 w dK=2Dk^2\int_K\varphi_2 u dK-D\int_K\nabla\varphi_2\cdot\mathbf{z}dK+D\int_{\partial K}\varphi_2(\hat{\mathbf{z}}\cdot\vec{\mathbf{n}})ds, \tag{2.19}$$

$$\int_K \boldsymbol{\theta}_2\cdot\mathbf{z}dK=-\int_K\nabla\boldsymbol{\theta}_2\cdot u dK+\int_{\partial K}\boldsymbol{\theta}_2(\hat{u}\cdot\vec{\mathbf{n}})ds. \tag{2.20}$$

Thus, the semi-discrete scheme corresponding to equations (2.17)-(2.20) is:

$$\int_{K^-} \varphi_{1,h}\partial_t u_h(t)dK = -\int_{K^-} \varphi_{1,h}\boldsymbol{\mu}\left(u_h(t)\right)dK - Dk^4\int_{K^-} \varphi_{1,h}u_h(t)dK + \int_{K^-}\left(\nabla\varphi_{1,h}\right)\cdot\mathbf{v}_h(t)dK$$
$$-\int_{\partial K^-}\left(\varphi_{1,h}^-\right)\left[\{|\mathbf{v}_h|\}\cdot\overrightarrow{\mathbf{n}}_{T^-} + \frac{\xi}{h_{T^-}}\left[\left[\mathbf{v}_h(t)\right]\right]\cdot\overrightarrow{\mathbf{n}}_{T^-}\right]ds, \tag{2.21}$$

$$\int_{K^-}\boldsymbol{\theta}_{1,\mathbf{h}}\cdot\mathbf{v}_h(t)dK = -\int_{K^-}\left(\nabla\boldsymbol{\theta}_{1,h}\right)\cdot w_h(t)dK$$
$$+\int_{\partial K^-}\left(\boldsymbol{\theta}_{1,\mathbf{h}}\right)\left[\{|w_h|\}\cdot\overrightarrow{\mathbf{n}}_{T^-} + \frac{\xi}{h_{T^-}}\left[\left[w_h(t)\right]\right]\cdot\overrightarrow{\mathbf{n}}_{T^-}\right]ds, \tag{2.22}$$

$$\int_{K^-}\varphi_{2,h}w_h(t)dK = 2Dk^2\int_{K^-}\varphi_{2,h}u_h(t)dK - D\int_{K^-}\nabla\varphi_{2,h}\cdot\mathbf{z}_h(t)dK + D\int_{\partial K^-}\varphi_{2,h}\left[\{|\mathbf{z}_h|\}\cdot\overrightarrow{\mathbf{n}}_{T^-}\right.$$
$$\left.+\frac{\xi}{h_{T^-}}\left[\left[\mathbf{z}_h(t)\right]\right]\cdot\overrightarrow{\mathbf{n}}_{T^-}\right]ds, \tag{2.23}$$

$$\int_{K^-}\boldsymbol{\theta}_{2,\mathbf{h}}\cdot\mathbf{z}_h(t)dK = -\int_{K^-}\nabla\boldsymbol{\theta}_{2,\mathbf{h}}\cdot u_h(t)dK$$
$$+\int_{\partial K^-}\boldsymbol{\theta}_{2,\mathbf{h}}\left[\{|u_h|\}\cdot\overrightarrow{\mathbf{n}}_{T^-} + \frac{\xi}{h_{T^-}}\left[\left[u_h(t)\right]\right]\cdot\overrightarrow{\mathbf{n}}_{T^-}\right]ds. \tag{2.24}$$

Let the local solution for the unknown functions be:

$$u_h(\mathbf{x},t)|_{K_r} = \sum_{j=1}^{N} U_{rj}(t)\boldsymbol{\phi}_{rj}(x), \tag{2.25}$$

$$\mathbf{v}_h(\mathbf{x},t)|_{K_r} = \sum_{j=1}^{N}\begin{bmatrix} V_{rj}^1(t) \\ V_{rj}^2(t)\end{bmatrix}\boldsymbol{\phi}_{rj}(x), \tag{2.26}$$

$$w_h(\mathbf{x},t)|_{K_r} = \sum_{j=1}^{N} W_{rj}(t)\boldsymbol{\phi}_{rj}(x), \tag{2.27}$$

$$\mathbf{z}_h(\mathbf{x},t)|_{K_r} = \sum_{j=1}^{N}\begin{bmatrix} Z_{rj}^1(t) \\ Z_{rj}^2(t)\end{bmatrix}(t)\boldsymbol{\phi}_{rj}(x). \tag{2.28}$$

Now, we have:

$$\sum_{j=1}^{N}\partial_t U_{rj}(t)\int_{K_r}\phi_{ri}(x)\phi_{rj}(x)dK = -\int_{K_r}\phi_{ri}(x)\mu\left(\sum_{j=1}^{N}U_{rj}(t)\phi_{rj}(x)\right)dK$$

$$-Dk^4\sum_{j=1}^{N}U_{rj}(t)\int_{K_r}\phi_{ri}(x)\phi_{rj}(x)dK + \sum_{j=1}^{N}\sum_{m=1}^{2}V_{rj}^{m}(t)\int_{K_r}(\partial_{x^m}\phi_{ri}(x))\phi_{rj}(x)dK$$

$$-\int_{\partial K_r}\phi_{r^-i}(x)\left[\frac{1}{2}\sum_{m=1}^{2}\mathbf{n}_{r^-}^{m}\left\{\sum_{j=1}^{N}V_{r^-j}^{m}(t)\phi_{r^-i}(x) + \sum_{j=1}^{N}V_{r^+j}^{m}(t)\phi_{r^+i}(x)\right\}\right.$$

$$\left.+\frac{\eta}{h_{K_{r^-}}}\left\{\sum_{j=1}^{N}W_{r^-j}^{m}(t)\phi_{r^-i}(x) + \sum_{j=1}^{N}W_{r^+j}^{m}(t)\phi_{r^+i}(x)\right\}\right]ds, \tag{2.29}$$

$$\sum_{j=1}^{N}V_{rj}^{m}(t)\int_{K_r}\phi_{ri}(x)\phi_{rj}(x)dK = -\sum_{j=1}^{N}W_{rj}(t)\int_{K_r}(\partial_{x^m}\phi_{ri}(x))\phi_{rj}(x)dK$$

$$+\frac{1}{2}\int_{\partial K_r}\phi_{r^-i}(x)\mathbf{n}_{r^-}^{m}\left[\sum_{j=1}^{N}W_{r^-j}(t)(t)\phi_{r^-j}(x)\right.$$

$$\left.+\sum_{j=1}^{N}W_{r^+j}(t)(t)\phi_{r^+j}(x)\right]ds,\ m=\{1,2\}, \tag{2.30}$$

$$\sum_{j=1}^{N}W_{rj}(t)\int_{K_r}\phi_{ri}(x)\phi_{rj}(x)dK = 2Dk^2\sum_{j=1}^{N}U_{rj}(t)\int_{K_r}\phi_{ri}(x)\phi_{rj}(x)dK$$

$$-D\sum_{j=1}^{N}\sum_{m=1}^{2}Z_{rj}^{m}(t)\int_{K_r}(\partial_{x^m}\phi_{ri}(x))\phi_{rj}(x)dK$$

$$+D\int_{\partial K_r}\phi_{r^-i}(x)\left[\frac{1}{2}\sum_{m=1}^{2}\mathbf{n}_{r^-}^{m}\left\{\sum_{j=1}^{N}Z_{r^-j}^{m}(t)\phi_{r^-i}(x) + \sum_{j=1}^{N}Z_{r^+j}^{m}(t)\phi_{r^+i}(x)\right\}\right.$$

$$\left.+D\frac{\eta}{h_{K_{r^-}}}\left\{\sum_{j=1}^{N}U_{r^-j}^{m}(t)\phi_{r^-i}(x) + \sum_{j=1}^{N}U_{r^+j}^{m}(t)\phi_{r^+i}(x)\right\}\right]ds, \tag{2.31}$$

$$\sum_{j=1}^{N} Z_{rj}^{m}(t)\int_{K_r}\phi_{ri}(x)\phi_{rj}(x)dK = -\sum_{j=1}^{N} U_{rj}(t)\int_{K_r}(\partial_{x^m}\phi_{ri}(x))\phi_{rj}(x)dK$$

$$+\frac{1}{2}\int_{\partial K_r}\phi_{r^-i}(x)\mathbf{n}_{r^-}^{m}\left[\sum_{j=1}^{N} U_{r^-j}(t)\phi_{r^-j}(x)\right.$$

$$\left.+\sum_{j=1}^{N} U_{r^+j}(t)\phi_{r^+j}(x)\right]ds, m=\{1,2\}. \tag{2.32}$$

Finally, the following nonlinear system of ODEs can be driven:

$$\mathbf{A}\frac{d\mathbf{X}}{dt} = \mathbf{B}\mathbf{X}(t) + \mathbf{F}(\mathbf{X}(t)), \tag{2.33}$$

that should be solved using an efficient algorithm. Now, we use an ETD scheme (Asante-Asamani *et al.*, 2016) for solving equation (2.33). Consider the following initial boundary value problem:

$$\begin{cases} u_t + Au = f(t,u), & in \quad \Omega, \quad t \in (0,T), \\ u(0,\cdot) = u_0, \end{cases} \tag{2.34}$$

in which:

- Ω is a Banach space.
- $-A$ generates an analytic semigroup E(t) = e^{-At} in Ω.
- f is an sufficiently smooth nonlinear term.
- $A : \mathcal{D}(A) \to \Omega$.

The proposed method in Asante-Asamani *et al.* (2016) is based on finding a numerical solution for the following integral form:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s,u(s))ds, \qquad \forall t \in [0,T]. \tag{2.35}$$

Now, the following recurrence relation can be concluded (Asante-Asamani *et al.*, 2016):

$$u(t_{n+1}) = e^{-Ak}u(t_n) + \int_{t_n}^{t_{n+1}} e^{-(t_{n+1}-s)A}f(s,u(s))ds. \tag{2.36}$$

Let $s = t_n + \tau k$ that $t_n = nk$ for $0 \le k \le k_0$, $0 \le n \le N$ and $\tau \in [0, 1]$ then we can rewrite equation (2.36) as follows (Asante-Asamani *et al.*, 2016):

$$u(t_{n+1}) = e^{-Ak}u(t_n) + k\int_0^1 e^{-Ak(1-\tau)}f(t_n+\tau k, u(t_n+\tau k))d\tau. \tag{2.37}$$

The above result is a basic issue in ETD scheme. Finally, the ETD scheme is as follows (Asante-Asamani *et al.*, 2016):

$$u_{n+1} = \left(I+\frac{1}{3}Ak\right)^{-1}\left[9u_n + 2kf(t_n,u_n) + kf\left(t_{n+1},u^*\right)\right] \tag{2.38}$$

$$+\left(I+\frac{1}{4}Ak\right)^{-1}\left[-8u_n - \frac{3k}{2}f(t_n,u_n) - \frac{k}{2}f\left(t_{n+1},u^*\right)\right],$$

$$u^* = (I+Ak)^{-1}\left[u_n + kf(u_n)\right]. \tag{2.39}$$

Now, we solve equation (2.33) using the above algorithm.

3. Proper orthogonal decomposition (POD) method

The proper orthogonal decomposition (POD) method is one of the reduced order methods (ROM) (Berkooz *et al.*, 1993; Everson and Sirovich, 1995; Kerschen *et al.*, 2005). The POD technique produces a new set of orthogonal basis function to apply in the numerical methods such as finite difference, finite element and finite volume. The POD technique can be found in several research papers for solving different physical models The POD technique is considered by many scholars (Chaturantabut, 2009; Chaturantabut and Sorensen, 2012; Fang *et al.*, 2009; Lin *et al.*, 2017; Ravindran, 2000a; Ravindran, 2000b; Ştefănescu and Navon, 2013; Ştefănescu *et al.*, 2014; Xiao *et al.*, 2015b). The POD approach has been used to solve the multi-species host-parasitoid system (Dimitriu *et al.*, 2015), compressible fluid and fractured solid (Fang *et al.*, 2009; Ravindran, 2000a; Ravindran, 2000b; Xiao *et al.*, 2017), Pacific Ocean model (Cao *et al.*, 2007), shallow water model (Ştefănescu and Navon, 2013; Ştefănescu *et al.*, 2014), 2D Burgers equation (Wang *et al.*, 2016), multiphase porous media flows (Xiao *et al.*, 2015b), Navier–Stokes equations (Xiao *et al.*, 2014), fluid-structure interaction (FSI) (Xiao *et al.*, 2013), dynamic PDEs based on the Smolyak sparse grid collocation (Xiao *et al.*, 2015a), transient heat conduction problems (Zhang and Xiang, 2015), convection-diffusion problems (Zhang *et al.*, 2016) and incompressible Navier–Stokes equation (Dehghan and Abbaszadeh, 2016a; Du *et al.*, 2012; Du *et al.*, 2013; Luo *et al.*, 2008; Xiao *et al.*, 2015b).

Consider the following difference scheme:

$$\mathcal{M}\mathbf{U}^{n+1} = \mathcal{N}\mathbf{U}^k + \mathcal{F}^k, \tag{3.1}$$

in which $\mathcal{M}$, $\mathcal{N}$, $\mathcal{F}$ and U^k denote the coefficients matrices, source term and the solution at the k-th step. Let U_{snap} be the snapshots matrix (Zhang and Xiang, 2015):

$$\mathbf{U}_{snap} = \left[\mathbf{U}_{n_1}\mathbf{U}_{n_2}\ldots\mathbf{U}_{n_d}\right]_{m\times d}. \tag{3.2}$$

Applying the SVD method for matrix U_{snap}, results (Zhang and Xiang, 2015):

$$\mathbf{U}_{snap} = \mathbf{U}_{m\times m}\begin{pmatrix} \sum_r & 0 \\ 0 & 0 \end{pmatrix}\mathbf{V}^T_{d\times d}, \tag{3.3}$$

in which:

$$\sum_r = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r),$$

and matrix $\mathbf{U}_{m\times m} = (\mathbf{ev}_1, \mathbf{ev}_2, \ldots \mathbf{ev}_m)$, is the orthogonal eigenvectors of $\mathbf{U}_{\mathrm{snap}}\mathbf{U}^T_{\mathrm{snap}}$. Now, we set (Zhang and Xiang, 2015):

$$\varpi_l = \left(U_1^{k,n_1}, U_2^{k,n_2}, \ldots, U_m^{k,n_l}\right), \quad l = 1, 2, \ldots, d. \tag{3.4}$$

The projection $\mathcal{P}_q$ is defined as (Zhang and Xiang, 2015):

$$\mathcal{P}_q(\varpi_l) = \sum_{i=1}^{q}(\mathbf{ev}_i, \varpi_l)\mathbf{ev}_i, \tag{3.5}$$

in which $q \leq d$. According to (Luo *et al.*, 2007):

$$\|\varpi_l - \mathcal{P}_q(\varpi_l)\|_2 \leq \sigma_{q+1}. \tag{3.6}$$

Thus, $\mathbf{ev}_1$, $\mathbf{ev}_2, \ldots$ $\mathbf{ev}_m$ represent the optimal POD basis. Thus, we put (Zhang and Xiang, 2015):

$$\mathbf{M} = [\mathbf{ev}_1, \mathbf{ev}_2, \ldots, \mathbf{ev}_q]. \tag{3.7}$$

Applying the new basis $\mathbf{M}$ to equation (3.1), yields:

$$\widehat{\mathcal{M}}\mathbf{U}^{n+1} = \widehat{\mathcal{N}}\mathbf{U}^k + \widehat{\mathcal{F}}^k, \tag{3.8}$$

in which:

$$\widehat{\mathcal{M}} = \mathbf{M}^T\mathcal{M}\mathbf{M}, \quad \widehat{\mathcal{N}} = \mathbf{M}^T\mathcal{N}\mathbf{M}, \quad \widehat{\mathcal{F}^k} = \mathbf{M}^T\mathcal{F}^k, \quad \widehat{\mathbf{U}^k} = \mathbf{M}^T\mathbf{U}^k, \tag{3.9}$$

respectively, and also $\widehat{\mathbf{U}}^0 = \mathbf{M}^T\mathbf{U}^0$. Thus, the new difference scheme is reduced to q elements. To calculate the energy of the snapshot data, we use the following term (Buchan *et al.*, 2015; Wang *et al.*, 2016)

$$I = \left(\sum_{i=1}^{d}\sigma_i\right)\left(\sum_{i=1}^{r}\sigma_i\right)^{-1}, \qquad d = \{1, 2, \ldots, r\}. \tag{3.10}$$

Theorem 3.1. (Luo *et al.*, 2013; Luo *et al.*, 2012) Let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_l > 0$ be the positive eigenvalues of A and $\vartheta^1, \vartheta^2, \ldots, \vartheta^l$ be the associated orthonormal eigenvalues. Then, the elements of POD basis of rank $d \leq l$ can be defined as:

$$\Phi_i = \frac{1}{\sqrt{L\lambda_i}} \sum_{j=1}^{L} \vartheta_j^i v_j, \qquad 1 \le i \le d \le l. \tag{3.11}$$

Also, the following error formula holds:

$$\frac{1}{L}\sum_{i=1}^{L} \|v_i - \sum_{j=1}^{d} (v_i, \Phi_i)_\omega \Phi_j\|_\omega^2 = \sum_{j=d+1}^{l} \lambda_j. \tag{3.12}$$

4. Investigation of numerical results

We use the explained numerical procedure for solving equation (1.1). We performed our computations using the MATLAB 2017 b software on an Intel Core i7 machine with 32 GB of memory.

The computational order of the developed method is checked by using the method of reference solution.

4.1 Test problem 1

The Swift–Hohenberg equation is (Gomez and Nogueira, 2012a):

$$\frac{\partial u}{\partial t} = -\mu(u) - Dk^4 u - \nabla^2(2Dk^2 u + D\nabla^2 u), \qquad \text{in} \quad \Omega \times (0, T), \tag{4.1}$$

$$\frac{\partial}{\partial n}(2Dk^2 u + D\nabla^2 u) = 0, \qquad \text{on} \ \ \Gamma \times [0, T], \tag{4.2}$$

$$\frac{\partial u}{\partial n} = 0, \text{on} \ \ \Gamma \times [0, T], \tag{4.3}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \text{in} \quad \overline{\Omega}. \tag{4.4}$$

In this model u is the scalar phase variable (Gomez and Nogueira, 2012a).

We solve this problem using the LDG method. We use initial guess s.t. if $x_1 < x < x_2$ then $u(x, y, 0) = 1$ and else $u(x, y, 0) = 0$ (Gomez and Nogueira, 2012a) in which:

$$x_1 = \sin\left(\frac{2\pi}{10} y\right) + 15, \qquad x_2 = \cos\left(\frac{2\pi}{10} y\right) + 25. \tag{4.5}$$

Figure 1 demonstrates the RMSE and the singular values (SVs) using 500 and 1,000 snapshots with $\tau = 10^{-4}$ and $h = 1/100$ for Test problem 1.

By computing the singular values in Figure 1, we can conclude that $\lambda_2 0 \le 5.31 \times 10^{-10}$ for step size $h = 1/100$. Furthermore, from equation (3.10), we find that $I(1) = 0.99842$ and $I(7) = 0.999998$. From Figure 1, we conclude $\lambda_{20} \le 5.31 \times 10^{-10}$ then in this example, we can chose 20 POD basis.

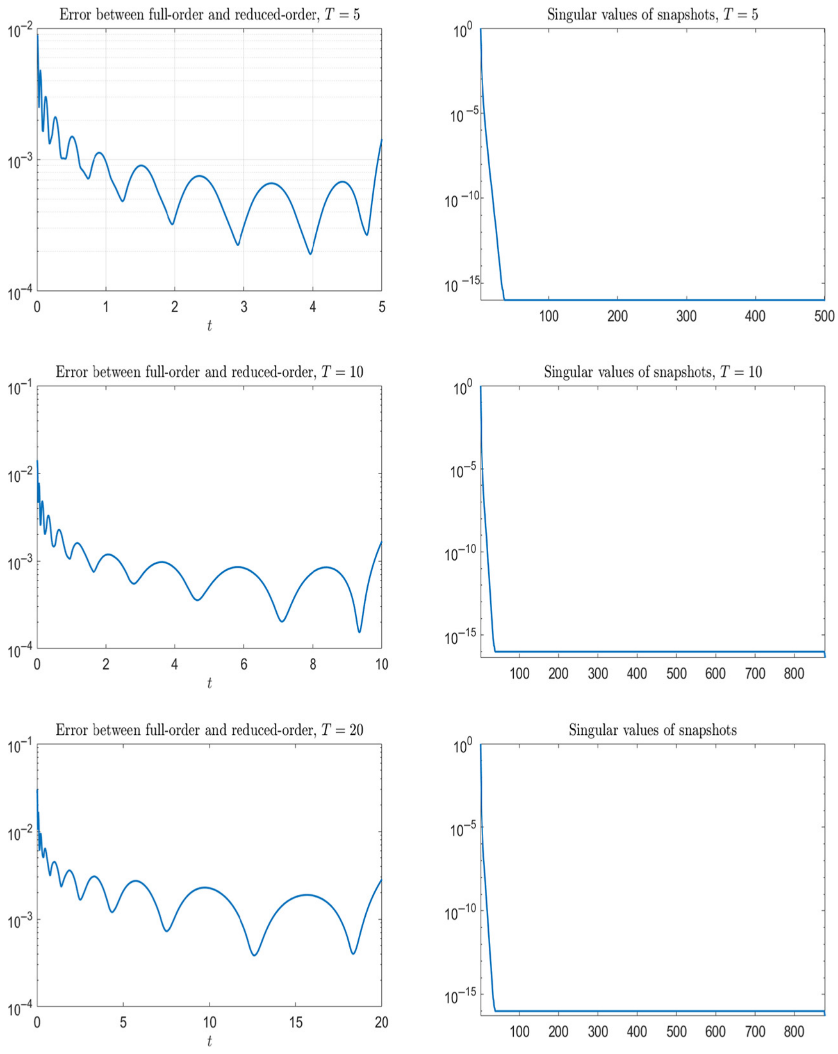

Figure 1.
RMSE using 500 and 1,000 snapshots with $h = 1/100$ and $\tau = 10^{-4}$ (left panel) and the singular values (right panel) for Test problem 1

Figure 2 shows the numerical solutions for Test problem 1 based on $D = 1$, $k = 1$, $g = 0$, $\epsilon = 2$, $h = 1/2$ and $\tau = 0.01$. Figure 3 presents the numerical simulation with $D = 1$, $k = 1$, $g = 0.5$, $\epsilon = 2$, $h = 1/2$ and $\tau = 0.01$ for Test problem 1. Also, errors and computational orders obtained for the present method for Test problem 1 are reported in Table I.

Table I presents the errors and computational orders obtained for present method for Test problem 1. Table II shows the used CPU time with $D = 1$, $k = 1$, $g = 0.5$, $\epsilon = 2$ and $\tau = 10^{-4}$.

4.2 Test problem 2

For the next example, we consider the following model (Klapp and Ovando, 2014):

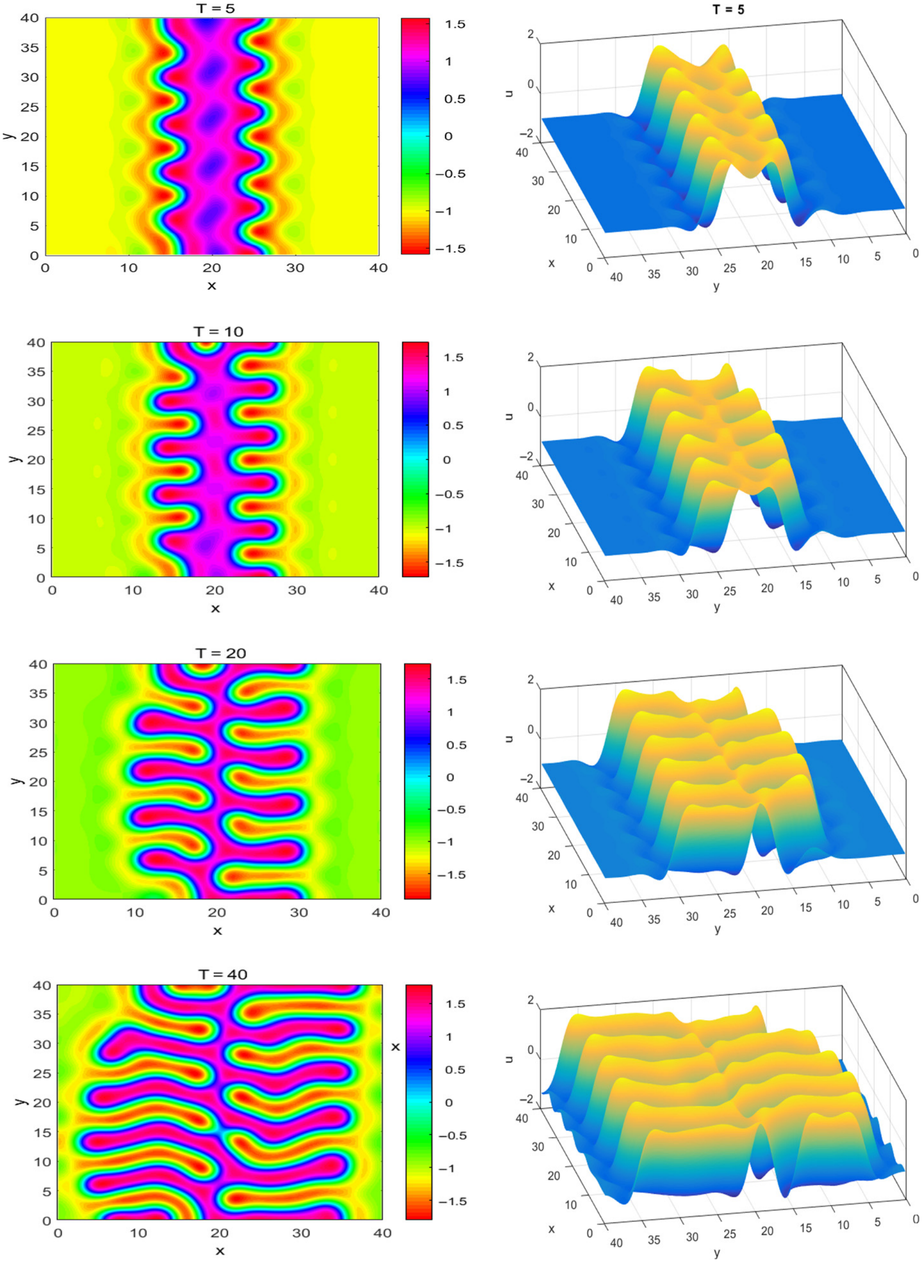

Figure 2.
Approximation solution with $D = 1$, $k = 1$, $g = 0$, $\epsilon = 2$, $h = 1/2$ and $\tau = 0.01$ for Test problem 1

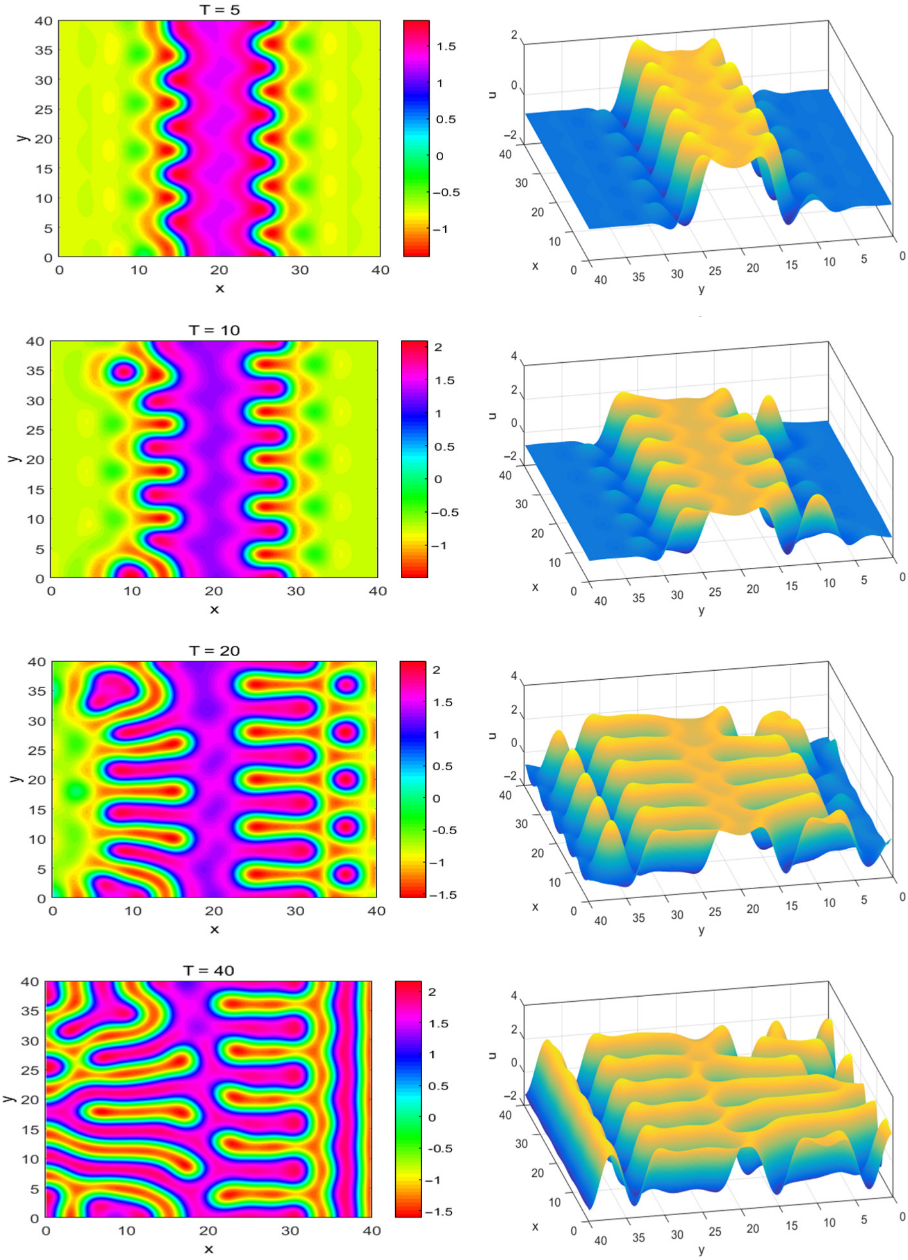

Figure 3.
Numerical simulations with $D = 1, k = 1, g = 0.5$, $\epsilon = 2, h = 1/2$ and $\tau = 0.01$ for Test problem 1

Table I. Results based on the present method for Test problem 1

τ	$h=1/4$ L_∞	C_1-order	$h=1/8$ L_∞	C_1-order
1/10	9.0913×10^{-1}	–	8.9791×10^{-2}	–
1/20	6.0629×10^{-1}	0.5844	4.9590×10^{-2}	0.8565
1/40	3.7001×10^{-1}	0.7124	2.9257×10^{-2}	0.7612
1/80	2.0122×10^{-1}	0.8787	1.5894×10^{-2}	0.8803
1/160	1.0169×10^{-1}	0.9847	8.0409×10^{-3}	0.9830
1/320	4.8393×10^{-2}	1.0713	3.8421×10^{-3}	1.0655
1/640	2.0926×10^{-2}	1.2095	1.6676×10^{-3}	1.2041
1/1,280	7.0056×10^{-3}	1.5787	5.5936×10^{-4}	1.5759

Table II. CPU Time (s) created with $\tau = 10^{-4}$

h	Main model CPU time (s)	PODLDG-ROM 15 basis	20 basis	30 basis
1	187	11	17	25
1/2	342	19	28	46
1/4	633	26	39	71
1/8	1,139	34	58	102
1/10	2,307	57	87	188

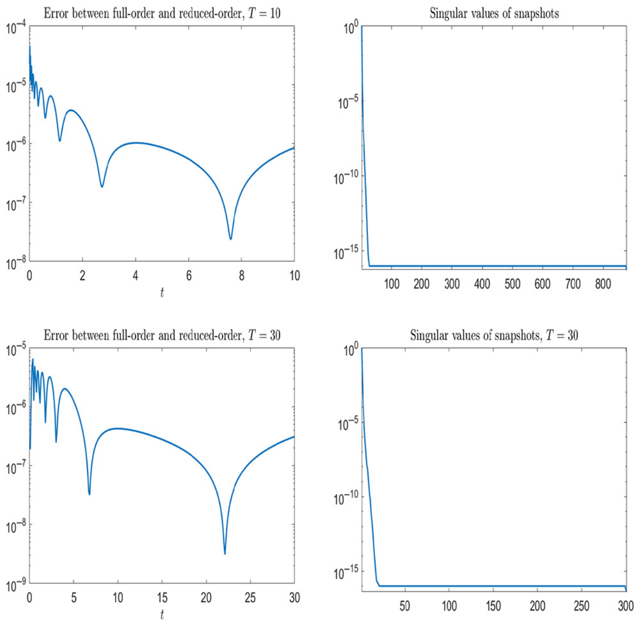

Figure 4. RMSE using on 500 and 1,000 snapshots with $g = 0.5$, $\epsilon = 0.05$, $h = 1$ and $\tau = 10^{-4}$ and the singular values for Test problem 2

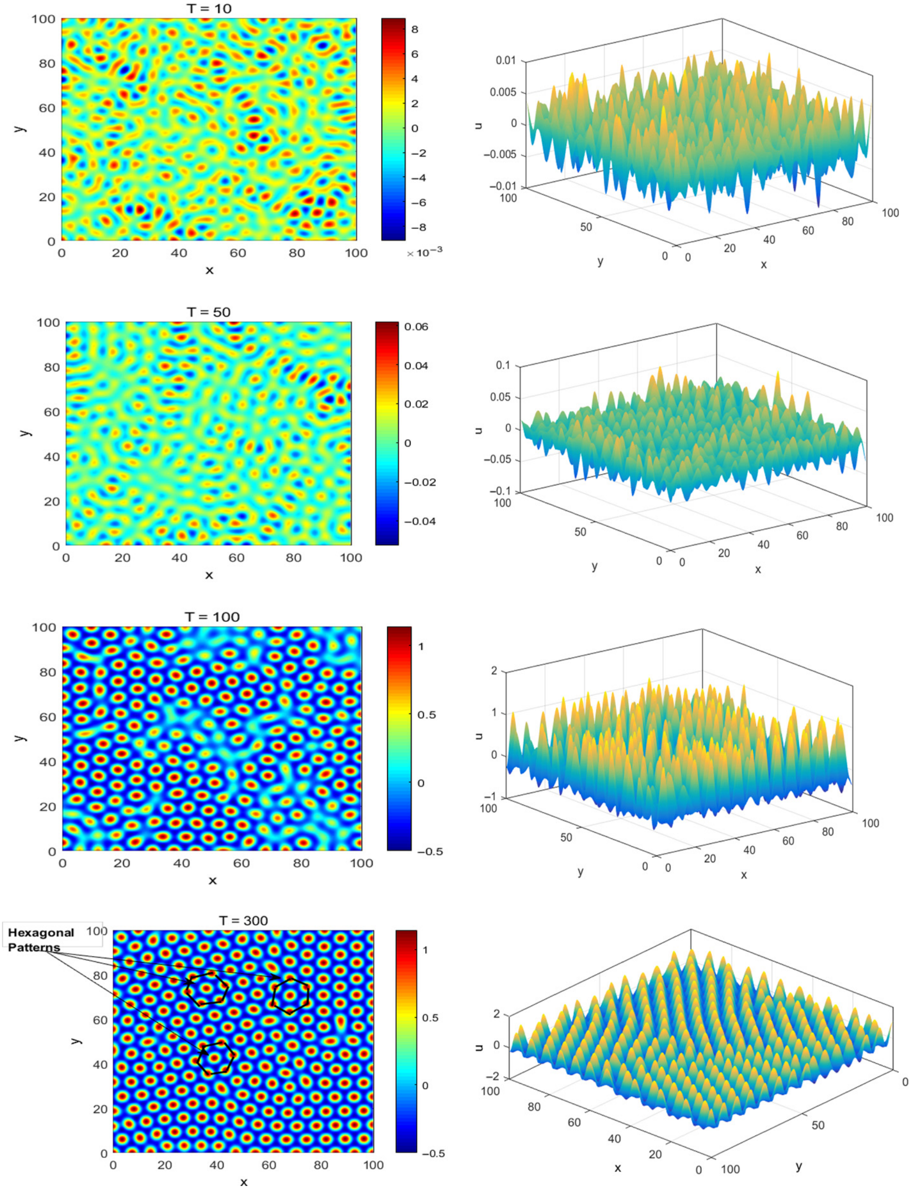

Figure 5.
Approximation solution with $g = 0.5$, $\epsilon = 0.05$, $h = 1$ and $\tau = 0.005$ for Test problem 2

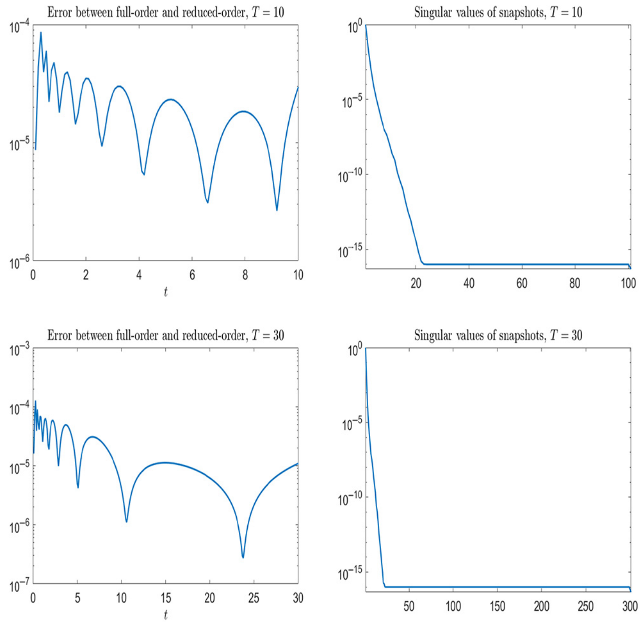

Figure 6. RMSE using 500 and 1,000 snapshots with $g = 0$, $\epsilon = 0.3$, $h = 1$ and $\tau = 10^{-4}$ and the singular values for Test problem 2

$$\frac{\partial u}{\partial t} = -(\Delta + 1)^2 u + \mu(u), \qquad \Omega \times (0, T), \tag{4.6}$$

in which:

$$\mu(u) = \varepsilon u + g u^2 - u^3, \tag{4.7}$$

based on the random initial guess and the periodic boundary condition.Figure 4 displays the RMSE using 500 and 1000 snapshots with $g = 0.5$, $\epsilon = 0.05$, $\tau = 10^{-4}$ and $h = 1$ (left plane) and the singular values (right panel).

According to Figure 4 and by computing the singular values, we can see $\lambda_1 5 \leq 4.37 \times 10^{-11}$ for step size $h = 1$. As well as Test problem 1, in the current example we use 15 POD basis associated to spatial size $h = 1$. Approximation solutions of Test problem 2 based on the $g = 0.5$, $\epsilon = 0.05$, $h = 1$ and $\tau = 0.005$ have been demonstrated in Figure 5.

Figure 6 illustrates the RMSE using 500 and 1,000 snapshots with $g = 0$, $\epsilon = 0.3$, $h = 1$ and $\tau = 10^{-4}$ (left plane) and the singular values (right panel) for Test problem 2. Figure 7 confirms that the hexagonal patterns are composed in $T = 300$. Furthermore, the approximation solutions of Test problem 2 with $\epsilon = 0.3, g = 0, h = 1$ and $\tau = 0.005$ have been depicted in Figure 8.

5. Conclusion

In this article, we considered generalized Swift–Hohenberg equation as a nonlinear fourth-order partial differential equation. The LDG finite element approach is used for

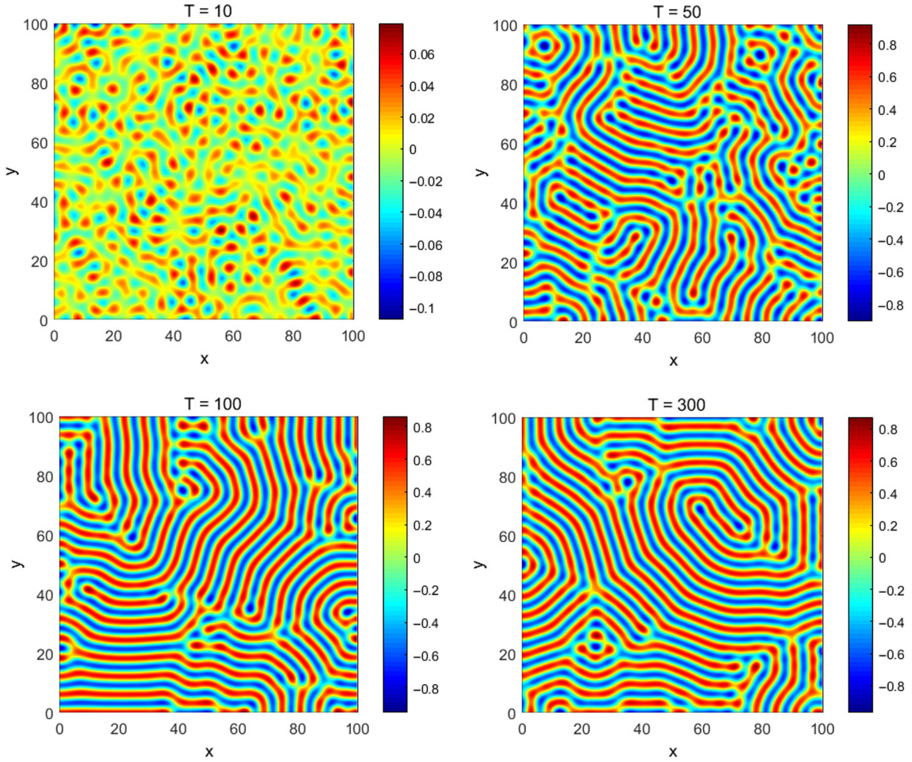

Figure 7.
Approximation solution based on the $g = 0$, $\epsilon = 0.3$, $h = 1$ and $\tau = 0.005$ for Test problem 2

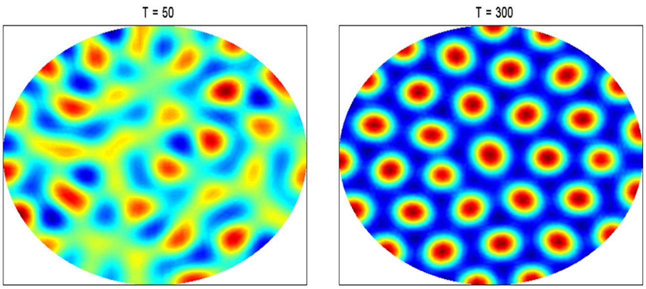

Figure 8.
Approximation solution based on the $g = 0$, $\epsilon = 0.3$, $h = 1$ and $\tau = 0.005$ on circular domain for Test problem 2

obtaining the numerical solutions of this model. First, the spatial direction has been discretized using the LDG finite element method and the energy stability for the semi-discrete LDG scheme has been proved. At the end of this process, a system of nonlinear ODEs has been achieved and to get the suitable and accurate results, an ETD scheme has been used. The developed algorithm has been examined on two different examples closed to the real problems. The achieved results acknowledge the susceptibility of the new numerical scheme.

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